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On the asymptotic distribution of GLR for impropriety of complex signals

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ABSTRACT

In this paper, the problem of testing impropriety (i.e., second-order noncircularity) of a sequence of complex-valued random variables (RVs) based on the generalized likelihood ratio test (GLRT) for Gaussian distributions is considered. Asymptotic (w.r.t. the data length) distributions of the GLR are given under the hypothesis that RVs are proper or improper, and under the true, not necessarily Gaussian distribution of the RVs. The considered RVs are independent but not necessarily identically distributed: assumption which has never been considered until now. This enables us to deal with the practical important situations of noncircular RVs disturbed by residual frequency offsets and additive circular noise. The receiver operating characteristic (ROC) of this test is derived as byproduct, an issue previously overlooked. Finally illustrative examples are presented in order to strengthen the obtained theoretical results.

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1. Introduction

For complex-valued RVs, many papers (see, e.g., [1–4]) show that significant performance gains can be achieved by second-order algorithms based on both $\mathbf{C}_x = E(\mathbf{x}\mathbf{x}^T)$ and $\mathbf{R}_x = E(\mathbf{x}\mathbf{x}^H)$. They exploit the statistical information contained in $\mathbf{C}_x$, provided it is nonzero in addition to that contained in the standard covariance matrix $\mathbf{R}_x$. These algorithms face an additional complexity. Moreover, some such algorithms (see e.g., [5]) adapted for improper or second-order noncircular signals, i.e., with nonzero matrices $\mathbf{C}_x$, fail or suffer of too slow convergence when they are used for proper or second-order circular signals. It is thus important to adapt the processing to the properness of the observation.

Hence, the question arises as to how we can classify a signal as proper or improper. This problem is a binary hypothesis test H_0: $\mathbf{C}_x = \mathbf{0}$ versus H_1: $\mathbf{C}_x \neq \mathbf{0}$. In practice, as the parameters $\mathbf{R}_x$ and $\mathbf{C}_x$ are clearly unknown, only the GLR detector can be used. This detector was introduced independently by Ollila and Koivunen [6] and Schreier et al. [7] under the traditional assumption of independent and identically distributed Gaussian samples $(\mathbf{x}_k)_{k=1,\ldots,K}$. But in these works, its performance was illustrated by a Monte Carlo simulation only. Walden and Rubin-Delanchy [8] derived recently this GLRT as well by formulating this testing problem in terms of real-valued Gaussian random vectors. Note that they have also presented a theoretical analysis of the null asymptotic distribution of the GLR with several numerical studies based on Monte Carlo simulations for the alternative distribution under the Gaussian distribution of the signals. Furthermore, there have been recent extensions of this GLRT to

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non-Gaussian RVs. Authors in [9] have extended this GLRT to complex elliptically symmetric distributions, with a slight adjustment by dividing it with an estimated scaled standardized fourth-order moment. Then in [10], a GLRT based on complex generalized Gaussian distributions have been provided. These extensions make the GLRT more robust to non-Gaussian distributions, but surprisingly they do not improve the performance for sub-Gaussian distributions [10], which include the majority of applications in communications and radar.

The aim of this paper is to complement the theoretical asymptotical analysis of [8,9]. The originality of our approach consists in considering the null and alternative asymptotic distribution of the GLR derived under the Gaussian distribution, but used in practice under independent not necessarily identically Gaussian distributed data. This paper is organized as follows. The GLRT is recalled for the convenience of the reader in Section 2. The asymptotic distribution of the GLR under the hypothesis that RVs are proper or improper is considered in Section 3, using the asymptotic distributions of the circularity coefficients given in [11]. This asymptotic distribution is given in the scalar case and then extended to the multidimensional case under the assumption of independent identically not necessarily Gaussian distributed RVs. An interpretable closed-form expression of the ROC is given in the scalar case due to the simplicity of the asymptotic distribution of the GLR. Then, extension of this study to independent nonidentically distributed RVs is considered in Section 4. This enables us to deal with practical situations of noncircular RVs disturbed by residual frequency offsets and additive circular noise. Finally some illustrative examples are presented in Section 5. Note that some results of this paper have been given in [12].

The following notations are used throughout the paper. Matrices and vectors are represented by bold upper case and bold lower case characters, respectively. Vectors are by default in column orientation, while T, H and $*$ stand for transpose, conjugate transpose, conjugate, respectively. $\mathrm{vec}(\cdot)$ is the "vectorization" operator that turns a matrix into a vector by stacking the columns of the matrix one below another which is used in conjunction with the Kronecker product $\mathbf{A}\otimes\mathbf{B}$ as the block matrix whose $(i,\ j)$ block element is $a_{i,j}\mathbf{B}$ and with the vec-permutation matrix $\mathbf{K}$ which transforms $\mathrm{vec}(\mathbf{C})$ to $\mathrm{vec}(\mathbf{C}^T)$ for any matrix $\mathbf{C}$.

2. Generalized likelihood ratio decision rule

We assume that $(\mathbf{x}_k)_{k=1,\ldots,K}\in\mathbb{C}^N$ is a realization of K independent identically zero-mean complex Gaussian distributed RVs. Their covariance matrices $\mathbf{R}_x=E(\mathbf{x}\mathbf{x}^H)$ and $\mathbf{C}_x=E(\mathbf{x}\mathbf{x}^T)$ are unknown. Consider the following binary composite hypothesis testing problem:

$$H_0:\mathbf{C}_x=\mathbf{0},\quad \mathbf{R}_x,$$

$$H_1:\mathbf{C}_x\neq\mathbf{0},\quad \mathbf{R}_x.$$

In the likelihood ratio, the GLR replaces the unknown parameters $\mathbf{R}_x$ and $\mathbf{C}_x$ by their maximum likelihood (ML) estimates. It is thus straightforward to derive its expression which is given by [6,7]

$$L(\mathbf{x},K)\stackrel{\text{def}}{=}\frac{p((\mathbf{x}_k)_{k=1,\ldots K};\hat{\mathbf{R}}_x,\hat{\mathbf{C}}_x,H_1)}{p((\mathbf{x}_k)_{k=1,\ldots K};\hat{\mathbf{R}}_x,\mathbf{0},H_0)}=\frac{\det(\hat{\mathbf{R}}_x)^K}{\det(\hat{\mathbf{R}}_{\tilde{x}})^{K/2}} \tag{1}$$

with $\hat{\mathbf{R}}_x\stackrel{\text{def}}{=}(1/K)\sum_{k=1}^K\mathbf{x}_k\mathbf{x}_k^H$ and $\hat{\mathbf{R}}_{\tilde{x}}\stackrel{\text{def}}{=}(1/K)\sum_{k=1}^K\tilde{\mathbf{x}}_k\tilde{\mathbf{x}}_k^H$ where $\tilde{\mathbf{x}}_k\stackrel{\text{def}}{=}[\mathbf{x}_k^T,\mathbf{x}_k^H]^T$. The GLRT decides H_1 if

$$L(\mathbf{x},K)>\lambda \tag{2}$$

and otherwise H_0. In the scalar case $N=1$, the GLRT is the UMP linearly invariant test [8]. But note that no uniformly most powerful (UMP) $\mathbb{C}$ linearly[2] invariant test for impropriety exists for $N>1$ [8]. It becomes especially simple

$$L(\mathbf{x},K)=(1-\hat{\gamma}_x^2)^{-K/2} \tag{3}$$

with $\hat{\gamma}_x=|(1/K)\sum_{k=1}^K x_k^2|/(1/K)\sum_{k=1}^K|x_k|^2$ is the ML estimate [13,11] of the circularity coefficient $\gamma_x\stackrel{\text{def}}{=}|E(x_k^2)|/E|x_k|^2$. By the increasing monotony of (3), the GLRT decides H_1 if

$$\hat{\gamma}_x>\lambda', \tag{4}$$

which is quite intuitive.

3. Asymptotic distribution of GLR for IID observations

Throughout this section, this GLRT is used for independent identically zero-mean nonnecessarily Gaussian distributed RVs $(\mathbf{x}_k)_{k=1,\ldots,K}$. For such non-Gaussian RVs, decision rule (2) is no longer a GLRT. However, it generally provides good performance in practice (see e.g., for the detection of a known signal corrupted by noncircular interference [14]) and is simple to implement.

3.1. Scalar complex random variable

Let x_k be a scalar valued RV of arbitrary distribution with finite fourth-order moments. We suppose that under H_0, x_k is circular up to the fourth-order.[3] Then, the following result is proved in the Appendix:

Result 1. Under the respective hypothesis H_0 and H_1, the following convergences in distribution hold when $K\to\infty$

$$\sqrt{\frac{K}{1+\frac{\kappa_x}{2}}}\hat{\gamma}_x\xrightarrow{\mathcal{L}}\mathcal{R}(1), \tag{5}$$

$$\sqrt{K}(\hat{\gamma}_x-\gamma_x)\xrightarrow{\mathcal{L}}\mathcal{N}(0,\sigma_\gamma^2)\quad\text{if }\gamma_x\neq 1. \tag{6}$$

In (5) and (6), $\mathcal{R}(1)$ and $\mathcal{N}(0,\sigma_\gamma^2)$ denote the Rayleigh distribution with unit scale (i.e., the chi distribution with two degrees of freedom χ_2) and the zero-mean Gaussian

[2] $\mathbb{C}$ linear transformations include rotation and scaling, but not widely linear operations.

[3] This means that not only $E(x_k^2)=0$, but also the fourth-order cumulants satisfy $\mathrm{cum}(x_k,x_k,x_k,x_k)=0$ and $\mathrm{cum}(x_k,x_k,x_k,x_k^*)=0$ [15]. We note, it is possible that $E(x_k^2)=0$ with $\mathrm{cum}(x_k,x_k,x_k,x_k)\neq 0$ or $\mathrm{cum}(x_k,x_k,x_k,x_k^*)\neq 0$. In this case, the asymptotic distribution of $\hat{\gamma}_x$ is much more involved (see the proof of Result 1 in the Appendix).

distribution with variance σ_γ^2, respectively, with

$$\sigma_\gamma^2=(1-\gamma_x^2)^2+\gamma_x^2\kappa_x+\frac{\kappa_x}{2}+\frac{\gamma_x^2\Re(\kappa'_x)}{2}-2\gamma_x^2\Re(\kappa''_x)\quad \text{if } \sigma_\gamma^2\neq 0, \tag{7}$$

where under H_0,κ_x is the normalized-like cumulant $\text{cum}(x_k,x_k,x_k^*,x_k^*)/(E(|x_k|^2))^2$, and under H_1,κ_x,κ'_x and κ''_x are the normalized-like cumulants $\text{cum}(x_k,x_k,x_k^*,x_k^*)/(E(|x_k|^2))^2$, $\text{cum}(x_k,x_k,x_k,x_k)/(E(x_k^2))^2$ and $\text{cum}(x_k,x_k,x_k,x_k^*)/E(|x_k|^2)\ E(x_k^2)$, respectively, which are invariant to any rotation of the distribution of x_k.

Naturally general expression (7) of σ_γ^2 simplifies for certain complex distribution classes for which the normalized-like cumulants κ_x, κ'_x and κ''_x are redundant. For example, the following result is proved in the Appendix.

Result 2. For generalized complex elliptically symmetric distributions (GCES)[4] introduced in [16] in the multidimensional case, σ_γ^2 (7) reduces to

$$\sigma_\gamma^2=(1-\gamma_x^2)^2\left(1+\frac{\kappa_x}{2+\gamma_x^2}\right). \tag{8}$$

Remark 1. This theoretical result means that the estimate $\hat{\gamma}_x$ is approximately Rayleigh (of scale $(1+\kappa_x/2)/K$) or Gaussian $\mathcal{N}(\gamma_x,\sigma_\gamma^2/K)$ distributed under H_0 and H_1, respectively, for $K\gg 1$. Furthermore the domain of validity of this approximation depends on γ_x and σ_γ^2 through the approximate relation $\gamma_x-2\sigma_\gamma/\sqrt{K}>0$. For practical use of this result, i.e., for probability of detection $P_D\neq 1$ and probability of false alarm $P_{FA}\neq 0$, note that the distribution of $\hat{\gamma}_x$ under H_0 and H_1 must overlap. This is roughly achieved for $\gamma_x-2\sigma_\gamma/\sqrt{K}<4\sqrt{1+\kappa_x/2}/\sqrt{K}$ as illustrated in Fig. 1.

Remark 2. For rectilinear RVs, $\gamma_x=1$ and thus $x_k=r_ke^{i\phi}$ where r_k is a real-valued RV and with ϕ fixed. In this case, the circularity coefficient γ_x is perfectly estimated, i.e., $\hat{\gamma}_x=1$. Consequently, the detection problem is singular and for a threshold λ' close to 1, P_D and P_{FA} are equal to 1 and 0, respectively.

Remark 3. Note that σ_γ^2 can be zero with $\gamma_x<1$ (an example of such a situation is given in [11]). In this case, the sequence $K(\hat{\gamma}_x-\gamma_x)$ converges in distribution [17, Theorem B, p. 124] to a Hermitian form $\mathbf{r}^H\boldsymbol{\Omega}\mathbf{r}$, with $\mathbf{r}$ a two dimensional zero-mean complex Gaussian RV. The distribution of this Hermitian form is defined by the right hand side of (17). But our first-order analysis does not allow one to specify the matrix $\boldsymbol{\Omega}$.

Remark 4. Note that for γ_x close to zero and $K\gg 1$, $2\ln L(\mathbf{x},K)=-K\ln(1-\hat{\gamma}_x^2)\approx K\hat{\gamma}_x^2$. Furthermore for Gaussian distributed x_k, $\kappa_x=0$. In these conditions (5) gives

$$2\ln L(\mathbf{x},K)\xrightarrow{\mathcal{L}}\chi_2^2\quad \text{under } H_0. \tag{9}$$

This asymptotic property is consistent with the constant false alarm rate (CFAR) detector where the number 2 of degree of freedom of the chi-squared distribution is equal to the number of real-valued components of $c_x\stackrel{\text{def}}{=}E(x_k^2)$, given

[4] Which include the Gaussian distribution.

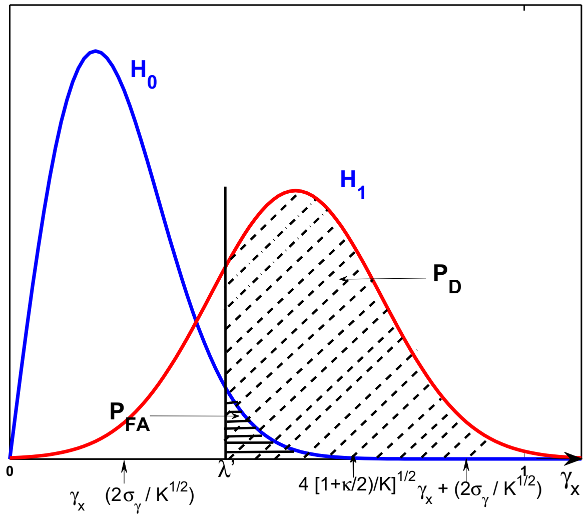

Fig. 1. Approximative probability density function (PDF) of $\hat{\gamma}_x$ under H_0 and H_1.

by Wilk's theorem [18, p. 132].[5] But for non-Gaussian distributions, detector (4) is no longer asymptotically CFAR. From the practical point of view, similarly as [9], by dividing the test statistic $\hat{\gamma}_x$ with $\sqrt{1+\hat{\kappa}_x/2}$ where $\hat{\kappa}_x$ is any consistent estimate of κ_x, we obtain an adjusted GLRT which becomes asymptotically CFAR. Hence, once the threshold is fixed for a given P_{FA}, the obtained P_D will depend naturally on the unknown parameters γ_x and σ_γ^2 (7).

Remark 5. For Gaussian distributed RVs, the normalized-like cumulants κ_x, κ'_x and κ''_x are zero. Thus the variance σ_γ^2 of the asymptotic distribution of $\hat{\gamma}_x$ under the hypothesis H_1 given by (7) and (8) becomes equal to $(1-\gamma_x^2)^2$. It is a decreasing function of γ_x. Consequently for a fixed P_{FA}, i.e., for fixed threshold λ', P_D is an increasing function of γ_x that does not depend on the power of x_k. This property is very intuitive.

For arbitrary, not necessarily Gaussian distributions of x_k, Result 1 allows us to derive

$$P_{FA}=P(\hat{\gamma}_x>\lambda'/H_0)\approx Q_{\chi_2^2}\left(\frac{K\lambda'^2}{1+\frac{\kappa_x}{2}}\right),$$

$$P_D=P(\hat{\gamma}_x>\lambda'/H_1)\approx Q_{\mathcal{N}}\left(\frac{\sqrt{K}(\lambda'-\gamma_x)}{\sigma_\gamma}\right),$$

where $Q_{\chi_2^2}(\cdot)$ and $Q_{\mathcal{N}}(\cdot)$ denote the complementary cumulative distribution functions (i.e., $Q_f(x)\stackrel{\text{def}}{=}\int_x^{+\infty}f(t)\,dt$ where $f(\cdot)$ is the associated probability density function) of the chi-squared distribution with 2 degrees of freedom and of the zero-mean, unit-variance Gaussian distribution, respectively, and where σ_γ is given by (7). Eliminating the threshold λ' between P_{FA} and P_D gives the following

[5] Note that this theorem has been used in [8,13] for vector and scalar cases to directly derive asymptotic distribution (9).

closed-form expression of the ROC of GLR detector (4)

$$P_D \approx Q_{\mathcal{N}}\left(\frac{\sqrt{\left(1+\frac{\kappa_x}{2}\right)Q^{-1}_{\chi^2_2}(P_{FA})}-\sqrt{K}\gamma_x}{\sigma_\gamma}\right). \quad (10)$$

From this expression, we clearly see that for fixed P_{FA}, P_D is an increasing function of the data length K and for Gaussian distributed RVs, an increasing function of the circularity coefficient γ_x.

3.2. Multidimensional complex random variable

In the multidimensional case ($N>1$), Result 1 cannot be easily extended as explained in the Appendix where we can only prove for arbitrary distributions with finite fourth-order moments the following result.

Result 3. Under hypothesis H_1, the following convergence in distribution holds when $K\to\infty$ for the decision statistic $\ell(\mathbf{x},K) \stackrel{\text{def}}{=} [L(\mathbf{x},K)]^{-2/K}$

$$\sqrt{K}(l(\mathbf{x},K)-\ell_1) \xrightarrow{\mathcal{L}} \mathcal{N}(0,\sigma_1^2) \quad \text{under } H_1, \quad (11)$$

where the expressions of ℓ_1 and σ_1^2 are derived in the Appendix.

Remark 6. Note that for the Gaussian distribution, i.e., for the only distribution for which the decision statistic $L(\mathbf{x},K)$ given by (1) is a GLR, Wilk's theorem [18, p. 132] applies[6] and gives

$$2\ln L(\mathbf{x},K) \xrightarrow{\mathcal{L}} \chi^2_{N(N+1)} \quad \text{under } H_0. \quad (12)$$

The degree of freedom of the chi-squared distributions is equal to the number $N(N+1)$ of real-valued independent parameters in the Hermitian matrix $\mathbf{C}_x$. Under H_1, in the particular case where $\mathbf{C}_x$ is "close" to $\mathbf{0}$ (see a more formal definition in [19, Chapter 23.7]), the analysis of [20, Section II] is valid and gives the following approximation of distribution when $K \gg 1$:

$$2\ln L(\mathbf{x},K) \stackrel{a}{\sim} \chi'_{N(N+1)}2(\mu) \quad \text{under } H_1.$$

In this expression, $\chi'_{N(N+1)}2(\mu)$ represents a noncentral chi-squared distribution with $N(N+1)$ degree of freedom and noncentral parameter μ. This parameter is a measure of the discrimination between H_0 and H_1. A general expression of this parameter which depends on K is given by [20, exp. (4)].

4. Extension to nonidentically distributed RVs

For practical purposes, RVs are not always identically distributed. In particular, when noncircular RVs are disturbed by residual frequency offsets and additive circular noise, RVs could be seen as circular depending on the signal to noise ratio (SNR) and the number K of samples. So in this section, we still consider the previous GLRT that has been derived under the assumption of independent identically zero-mean complex Gaussian distribution. But it is used here for independent zero-mean nonnecessarily identically Gaussian distributed RVs[7] $(x_k)_{k=1,\ldots,K}$. To take account of the dependence of the distribution of x_k with k, the following notation is used: $r_{x,k} \stackrel{\text{def}}{=} E|x_k^2|$, $c_{x,k} \stackrel{\text{def}}{=} E(x_k^2)$, $\overline{r}_{x,K} \stackrel{\text{def}}{=} (1/K)\sum_{k=1}^K r_{x,k}$, $\overline{c}_{x,K} \stackrel{\text{def}}{=} (1/K)\sum_{k=1}^K c_{x,k}$, $\text{cum}_{x,k} \stackrel{\text{def}}{=} \text{cum}(x_k,x_k,x_k^*,x_k^*)$, $\text{cum}'_{x,k} \stackrel{\text{def}}{=} \text{cum}(x_k,x_k,x_k,x_k)$ and $\text{cum}''_{x,k} \stackrel{\text{def}}{=} \text{cum}(x_k,x_k,x_k,x_k^*)$.

For arbitrary distributions with finite fourth-order moments such that the following Lyapunov conditions [21, Theorem 2.7.2] are satisfied[8]:

$$\lim_{K\to\infty}\frac{\sum_{k=1}^K E||x_k^2|-r_{x,k}|^3}{(\sqrt{\sum_{k=1}^K E((|x_k^2|-r_{x,k})^2)})^3}=0 \quad \text{and}$$
$$\lim_{K\to\infty}\frac{\sum_{k=1}^K E|x_k^2-c_{x,k}|^3}{(\sqrt{\sum_{k=1}^K E((x_k^2-c_{x,k})^2)})^3}=0, \quad (13)$$

where $r_{x,k}$, $c_{x,k}$, $\text{cum}_{x,k}$, $\text{cum}'_{x,k}$ and $\text{cum}''_{x,k}$ are bounded and where we suppose that under H_0, $(x_k)_{k=1,\ldots,K}$ are circular up to the fourth-order, the following result extending Result 1 is proved in the Appendix.

Result 4. Under the respective hypotheses H_0 and H_1, the following convergences in distribution hold when $K\to\infty$

$$\sqrt{\frac{K}{\alpha_K+\frac{\kappa_{x,K}}{2}}}\hat{\gamma}_x \xrightarrow{\mathcal{L}} \mathcal{R}(1), \quad (14)$$

$$\sigma^{-1}_{\gamma,K}(\hat{\gamma}_x-\gamma_{x,K}) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1), \quad (15)$$

where $\alpha_K \stackrel{\text{def}}{=} (1/\overline{r}^2_{x,K})(1/K)\sum_{k=1}^K r^2_{x,k}$ and $\kappa_{x,K} \stackrel{\text{def}}{=} (1/\overline{r}^2_{x,K})(1/K)\sum_{k=1}^K \text{cum}_{x,k}$, $\gamma_{x,K}$ is the time-averaged circularity coefficient $|\overline{c}_{x,K}|/\overline{r}_{x,K} = |(1/K)\sum_{k=1}^K E(x_k^2)|/(1/K)\sum_{k=1}^K E|x_k|^2$ and where the expression of $\sigma_{\gamma,K}$ is derived in the Appendix.

Remark 7. Clearly for identically distributed RVs, $r_{x,k} = \overline{r}_{x,K} = r_x$, $\text{cum}_{x,k}/r_x^2 = \kappa_x$ and thus $\alpha_K = 1$ and $\kappa_{x,K} = \kappa_x$ in (14) and Result 3 reduces to Result 1 under H_0. Under H_1, the derivation of $\sigma^2_{\gamma,K}$(23) in the Appendix comes down to the proof of (6), (7) given in [11] for identically distributed RVs where $\sigma_{\gamma,K} = \sigma_\gamma/\sqrt{K}$.

5. Illustrative examples

This section has two purposes. First, we examine the domain of validity of our asymptotic results, and second, we study the performance of the GLR detector in a specific example.

The following MIMO channel (extension of the example given in [7]) that transmits Q independent equiprobable BPSK symbols $a_{q,k} \in \{-1,+1\}$ over an additive noise channel is considered. It also rotates independently the phase of the transmitted symbols $a_{q,k}$ by $\phi_{q,k}$ and are

[6] Note that Wilk's theorem has been invoked in this context in [8,9].

[7] We only consider scalar complex-valued RVs, because the extension to multidimensional complex-valued RVs would involve overly too cumbersome notations.

[8] Which are not severe and are clearly satisfied for the RVs described by (16).

disturbed by residual frequency offsets Δf_q.

$$\mathbf{x}_k = \sum_{q=1}^{Q} \sigma_q a_{q,k} e^{i\phi_{q,k}} e^{i2\pi k \Delta f_q} \mathbf{s}_q + \mathbf{n}_k, \tag{16}$$

where σ_q and $\mathbf{s}_q$ are Q unknown amplitudes and steering vectors with unit first component. The components of $\mathbf{n}_k$ are independent zero-mean complex circular Gaussian RV of unknown variance σ_n^2.

We consider three experiments. In the first one, there is no residual frequency offset and under H_0 and H_1, we assume that the phase terms $(\phi_{q,k})_{k=1,..,K,q=1,..,Q}$ are independent and, respectively, uniformly distributed on $[0,2\pi]$ or Gaussian distributed with mean ϕ_{q_0} and variance $\sigma^2_{\phi_q}$. So we are interested in classifying this channel as either incoherent or partially coherent. This is a binary composite hypothesis testing problem. We easily deduce that

$$\mathbf{R}_x = \sum_{q=1}^{Q} \sigma_q^2 \mathbf{s}_q \mathbf{s}_q^H + \sigma_n^2 \mathbf{I}_Q$$

and

$$\mathbf{C}_x = \begin{cases} \mathbf{0} & \text{under } H_0, \\ \sum_{q=1}^{Q} \sigma_q^2 e^{2i\phi_{q0}} e^{-2\sigma^2_{\phi_q}} \mathbf{s}_q \mathbf{s}_q^T & \text{under } H_1. \end{cases}$$

For $Q=1$ and $N=1$, $\kappa_x=-1/(1+\rho_x^{-1})^2$ under H_0 and $\gamma_x = e^{-2\sigma^2_{\phi_1}}/(1+\rho_x^{-1})$, $\kappa_x=-(1+e^{-4\sigma^2_{\phi_1}})/(1+\rho_x^{-1})^2$ $\kappa'_x=e^{-4\sigma^2_{\phi_1}}-3$ and $\kappa''_x=-2/(1+\rho_x^{-1})$ under H_1, with an SNR of $\rho_x \stackrel{\text{def}}{=} \sigma_1^2/\sigma_n^2$.

Fig. 2 shows the detection performance P_D for different fixed P_{FA} for $N=Q=1$ as a function of the SNR for two values of σ_{ϕ_1} deduced from the asymptotic distribution of $\hat{\gamma}_x$ given by Result 1. We see that the P_D for fixed P_{FA} is very sensitive to the coherence of the channel. When σ_{ϕ_1} increases for a fixed SNR, the circularity coefficient γ_x decreases and detection worsens.

In the second experiment, model (16) with $N=Q=1$ is compared to the Gaussian model obtained when $\phi_{1,k}$ does not depend on k and $a_{1,k}$ are independent zero-mean complex circular or real-valued Gaussian RVs under H_0 and H_1, respectively. Fig. 3 shows the northwest corner of the ROC curve for the GLRT detector for $K=100$ and $\rho_x=0.63$ $(-2$ dB) for BPSK model with a coherent channel (i.e., $\sigma_{\phi_1}=0$) and Gaussian model, and thus associated with the same value of $\gamma_x=0.387$. We note that the ROC curve is sensitive to the distribution of the RVs x_k, the performance is improved for the BPSK model w.r.t. the Gaussian model and that the empirical ROC fits the asymptotic theoretical ROC for the relatively small data length $K=100$.

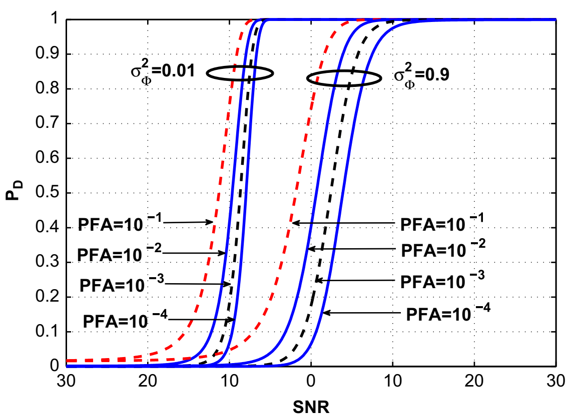

Fig. 2. P_D for four different fixed P_{FA} and two values of σ_{ϕ_1} as a function of SNR for $N=Q=1$ and $K=100$.

Fig. 4 shows the ROC curve for the GLRT detector for the same parameters as in Fig. 3, but with four residuals of frequency offset Δf_1 for which $r_{x,k}=\bar{r}_{x,K}=\sigma_1^2+\sigma_n^2$, $c_{x,k}=\sigma_1^2 e^{2i\phi_1}e^{4\pi i k\Delta f_1}$, $\bar{c}_{x,K}=\sigma_1^2 e^{2i\phi_1}e^{2\pi i(K-1)\Delta f_1}(\sin 2\pi K\Delta f_1/\sin 2\pi\Delta f_1)$,

$$\gamma_{x,K} = \frac{1}{1+\rho_x^{-1}} \frac{1}{K} \left| \frac{\sin 2\pi K \Delta f_1}{\sin 2\pi \Delta f_1} \right|$$

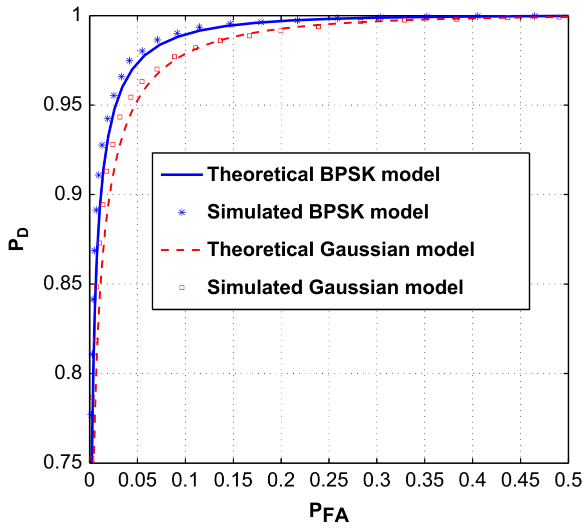

Fig. 3. Asymptotic theoretical and empirical (with 10 000 Monte Carlo runs) ROC curve associated with BPSK and Gaussian model.

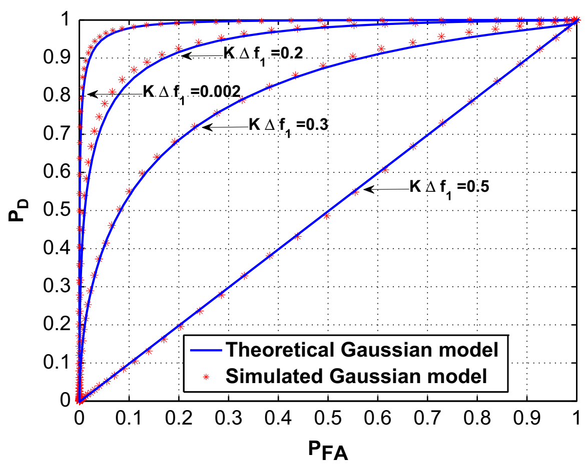

Fig. 4. Asymptotic theoretical and empirical (with 10 000 Monte Carlo runs) ROC curve associated with Gaussian model for four values of $K\Delta f_1$.

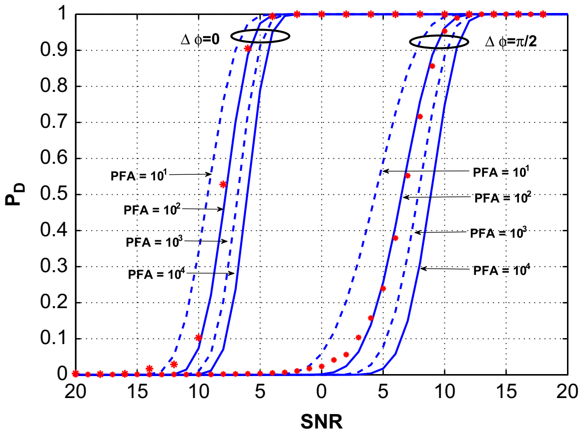

Fig. 5. Asymptotic theoretical and empirical (with 10 000 Monte Carlo runs) P_D for four different fixed P_{FA} and two values of $\Delta\phi$ as a function of SNR for $N=Q=2$ and $K=100$.

and $\text{cum}_{x,k}=0$ for a Gaussian signal. Comparing to Fig. 3, we note in Fig. 4 a degradation owing to the frequency offset for which the time-averaged circularity coefficient gets closer to zero under H_1. The performance of the detector begin decreasing from $K\Delta f_1=0.002$ for which $P_{FA}=0.1$ and 0.05 are obtained for $P_D=0.980$ and 0.960, respectively, against $P_D=0.989$ and 0.970 for no residual frequency offset. The detection capability collapses for $K\Delta f_1=0.5$ where the time-averaged circularity coefficient $\gamma_{x,K}=0$. We see also that the empirical ROC fits the asymptotic theoretical ROC for the relatively small data length $K=100$.

Finally in the third experiment, we consider the multi-dimensional Gaussian model[9] ($\phi_q \overset{\text{def}}{=} \phi_{q,k}$ does not depend on k and $a_{q,k}$ are independent zero-mean complex circular or real-valued Gaussian RVs under H_0 and H_1, respectively), with no residual of frequency offset. Here, $Q=2$, $\sigma_1=\sigma_2$, with an array of $N=2$ omnidirectional sensors equispaced half a wavelength apart. The direction of arrival with respect to broadside of the two sources are $\theta_1=0^\circ$ and $\theta_2=5^\circ$. Fig. 5 shows the detection performance P_D for different fixed P_{FA} as a function of the SNR for two values of $\Delta\phi \overset{\text{def}}{=} \phi_1-\phi_2$. P_D and P_{FA} are deduced from the asymptotic distribution of $l(\mathbf{x},K)$ under H_1 given by Result 4 and of $2\ln L(\mathbf{x},K)$ under H_0 given by (12), respectively. We see that the GLRT is very sensitive to $\Delta\phi$. In particular for very close DOAs (i.e., $\mathbf{s}_1 \approx \mathbf{s}_2$) and equi-powered sources, $\mathbf{C}\approx\mathbf{0}$ under H_1 for $\Delta\phi=\pi/2$ radians, which implies a very bad capability of circularity detection. Furthermore we see that the empirical P_D fits the asymptotic P_D for the relatively small data length $K=100$, except for weak P_D.

[9] We note that in this case under H_0, the asymptotic distribution of the test statistic is only available for Gaussian distributions of the RVs (see (12)). In this case, this test is asymptotically CFAR and once the threshold is fixed for a given P_{FA}, the obtained P_D derived by (11) will depend naturally on the unknown parameters $\ell_1=\det[\mathbf{I}-(\mathbf{R}_x^{-1}\mathbf{C}_x)^*\mathbf{R}_x^{-1}\mathbf{C}_x]$ and σ_1^2 derived in the Appendix.

6. Conclusion

In this paper, some new enlightening results about the asymptotic distribution of the GLR for impropriety of complex signals have been investigated. The associated GLRT derived under the usual assumption of independent identically distributed Gaussian RVs is studied under nonnecessarily identical Gaussian distributions of the RVs. For the scalar case, the asymptotic distribution of the circularity coefficient has been given under H_0 and H_1 for independent identical or independent nonidentical arbitrary distributions of the RVs. In particular this allows us to deal with the important practical situations where discrete RVs are disturbed by residual frequency offsets and additive Gaussian circular noise which has never been considered until now. For the multidimensional case, the asymptotic distribution of the GLR has been given under H_1 for independent and identically arbitrary distributions of the RVs. These results enable us to specify the probability of detection for a specified probability of false alarm, and thus to derive the ROC of this test, an issue previously totally overlooked.

Appendix

Proof of Result 1. Under H_1, (6) is directly issued from [11, Result 3]. But under H_0, [11, Result 3] is not valid because it does not holds for $\gamma_x=0$. Nevertheless the analysis of [11] still applies. The classical central limit[10] applied to the independent identically distributed bidimensional complex RVs $(\hat{r}_x,\hat{c}_x)$ with $\hat{r}_x=(1/K)\sum_{k=1}^{K}|x_k^2|$ and $\hat{c}_x=(1/K)\sum_{k=1}^{K}x_k^2$ yields for $\gamma_x=0$

$$\sqrt{K}\begin{pmatrix}\hat{r}_x-r_x\\ \hat{c}_x-c_x\end{pmatrix}\xrightarrow{\mathcal{L}}\mathcal{N}_C\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\sigma_x^4+\text{cum}_x & \text{cum}''^*_x\\ \text{cum}''_x & 2\sigma_x^4+\text{cum}_x\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}\sigma_x^4+\text{cum}_x & \text{cum}''_x\\ \text{cum}''_x & \text{cum}'_x\end{pmatrix}\right), \tag{17}$$

where $\sigma_x^2 \overset{\text{def}}{=} E|x_k^2|$, $\text{cum}_x \overset{\text{def}}{=} \text{cum}(x_k,x_k,x_k^*,x_k^*)$, $\text{cum}'_x \overset{\text{def}}{=} \text{cum}(x_k,x_k,x_k,x_k)$ and $\text{cum}''_x \overset{\text{def}}{=} \text{cum}(x_k,x_k,x_k,x_k^*)$. Then, considering the mapping

$$(\hat{r}_x,\hat{c}_x)\mapsto\hat{m}_x=\frac{\hat{c}_x}{\hat{r}_x}\mapsto\hat{\gamma}_x=|\hat{m}_x|, \tag{18}$$

whose differential of the first step is

$$dm=\frac{1}{r}dc \tag{19}$$

under H_0, the standard theorem of continuity (see e.g., [17, Theorem A, p. 122]) on regular functions of asymptotically Gaussian statistics applies. Consequently, we obtain the following convergence in distribution to a complex zero-mean Gaussian distribution of variance $(1/\sigma_x^4)(2\sigma_x^4+\text{cum}_x)$ and pseudo-variance $(1/\sigma_x^4)\text{cum}'_x$

$$\sqrt{K}(\hat{m}_x-0)\xrightarrow{\mathcal{L}}\mathcal{N}_C\left(0,2+\frac{\text{cum}_x}{\sigma_x^4},\frac{\text{cum}'_x}{\sigma_x^4}\right). \tag{20}$$

[10] $\mathcal{N}_C(\mathbf{m},\mathbf{R},\mathbf{C})$ denotes the complex Gaussian distribution with mean $\mathbf{m}$, and covariances $\mathbf{R}$ and $\mathbf{C}$.

This complex Gaussian distribution becomes circular ($\text{cum}'_x = 0$) for x_k circular up to the fourth-order. With $\hat{\gamma}_x = |\hat{m}_x|$, convergence in distribution (5) is proved. □

Proof of Result 2. From [16], the GCES distribution of x_k is defined in the scalar case from the distribution of the real-valued bivariate RV $(\Re(x_k),\Im(x_k))$ which is real elliptical symmetric (RES) distributed. In the zero-mean case, this RES distribution is defined as a linear transform in $\mathbb{R}^2$ of a spherically symmetric distribution [22]. Consequently as a linear transform in $\mathbb{R}^2$ is equivalent to an $\mathbb{R}$-linear transform in $\mathbb{C}$ [23], x_k is zero-mean GCES distributed, if there exist complex valued scalars a and b such that $x_k = au_k + bu_k^*$ where u_k is an arbitrary complex circular RV. Consequently the cumulants of x_k satisfy the following relations:

$$\text{cum}(x_k,x_k,x_k^*,x_k^*) = ((|a|^2+|b|^2)^2+2|a|^2|b|^2)\text{cum}(u_k,u_k,u_k^*,u_k^*),$$

$$\text{cum}(x_k,x_k,x_k,x_k) = 6a^2b^2\text{cum}(u_k,u_k,u_k^*,u_k^*),$$

$$\text{cum}(x_k,x_k,x_k,x_k^*) = 3ab(|a|^2+|b|^2)\text{cum}(u_k,u_k,u_k^*,u_k^*).$$

Using

$$E|x_k|^2 = (|a|^2+|b|^2)E|u_k|^2 \quad \text{and} \quad E(x_k^2) = 2abE|u_k|^2,$$

the normalized-like cumulants κ'_x and κ''_x become

$$\kappa'_x = \kappa''_x = \left(\frac{3}{2+\gamma_x^2}\right)\kappa_x.$$

Plugging these expressions in (7), gives expression (8) of Result 3. □

Proof of Result 3. With $\ell(\mathbf{x},K) = \det[\mathbf{I}-(\hat{\mathbf{R}}_x^{-1}\hat{\mathbf{C}}_x)^*\hat{\mathbf{R}}_x^{-1}\hat{\mathbf{C}}_x]$, from (1) where $\hat{\mathbf{C}}_x \stackrel{\text{def}}{=} (1/K)\sum_{k=1}^K \mathbf{x}_k\mathbf{x}_k^T$, the proof of Result 2 follows the same steps that for Result 1.

Deriving the asymptotic distribution of $\ell(\mathbf{x},K)$ under H_0 and H_1 is based on the following mapping:

$$(\hat{\mathbf{R}}_x,\hat{\mathbf{C}}_x) \mapsto \hat{\mathbf{M}}_x = \hat{\mathbf{R}}_x^{-1}\hat{\mathbf{C}}_x \mapsto \hat{\Sigma}_x = \hat{\mathbf{M}}_x^*\hat{\mathbf{M}}_x \mapsto \ell(\mathbf{x},K) = \det[\mathbf{I}-\hat{\Sigma}_x]. \tag{21}$$

Using the asymptotic Gaussian distribution of $(\hat{\mathbf{R}}_x,\hat{\mathbf{C}}_x)$ [3,11] derived from the classical central limit theorem, the differential of the different sub-mappings of (21), the chain rule and standard properties of the vec operator [24, Chapter 2.4], the standard theorem of continuity (see e.g., [17, p. 122]) on regular functions of asymptotically Gaussian statistics applies.

In particular under H_0, where $\mathbf{x}_k$ is circular up to the fourth-order, the differential of $\mathbf{M}_x$ at $(\mathbf{R}_x,\mathbf{C}_x)=(\mathbf{R}_x,\mathbf{0})$ is similar as (19), given by

$$d\mathbf{M}_x = -\mathbf{R}_x^{-1}\, d\mathbf{R}_x\mathbf{R}_x^{-1}\mathbf{C}_x + \mathbf{R}_x^{-1}\, d\mathbf{C}_x = \mathbf{R}_x^{-1}\, d\mathbf{C}_x,$$

$$\text{vec}(d\mathbf{M}_x) = (\mathbf{I}\otimes\mathbf{R}_x^{-1})\text{vec}(d\mathbf{C}_x). \tag{22}$$

Consequently, (20) becomes here

$$\sqrt{K}(\text{vec}(\hat{\mathbf{M}}_x)-\mathbf{0}) \xrightarrow{\mathcal{L}} \mathcal{N}_C(\mathbf{0},\mathbf{R}_M,\mathbf{C}_M)$$

with

$$\mathbf{R}_M = (\mathbf{I}\otimes\mathbf{R}_x^{-1})\mathbf{R}_C(\mathbf{I}\otimes\mathbf{R}_x^{-1}) \quad \text{and} \quad \mathbf{C}_M = (\mathbf{I}\otimes\mathbf{R}_x^{-1})\mathbf{C}_C(\mathbf{I}\otimes\mathbf{R}_x^{-*}),$$

where $\mathbf{R}_C$ and $\mathbf{C}_C$ are the covariance matrices of the asymptotic distribution of $\hat{\mathbf{C}}_x$ given [11] by[11]

$$\mathbf{R}_C = \mathbf{R}_x\otimes\mathbf{R}_x + \mathbf{K}(\mathbf{R}_x\otimes\mathbf{R}_x)+\mathbf{Q}_x \quad \text{and}$$
$$\mathbf{C}_C = \mathbf{C}_x\otimes\mathbf{C}_x + \mathbf{K}(\mathbf{C}_x\otimes\mathbf{C}_x)+\mathbf{Q}'_x,$$

for which here $\mathbf{C}_C=\mathbf{0}$. Consequently $\mathbf{C}_M=\mathbf{0}$ as in the scalar case, $\hat{\mathbf{M}}_x$ is still asymptotically circular Gaussian distributed under H_0 for $\mathbf{x}_k$ circular up to the fourth-order and the differential of the mapping $\hat{\mathbf{M}}_x \mapsto \hat{\Sigma}_x$ at $\mathbf{M}_x=\mathbf{0}$ is still zero. But in contrast to the scalar case, the derivation of the asymptotic distribution of $\hat{\Sigma}_x$ needs the second differential of this mapping, which is not accessible by our first-order analysis.

Under H_1, with the differential of the mapping $(\hat{\mathbf{R}}_x,\hat{\mathbf{C}}_x) \mapsto \hat{\mathbf{M}}_x$ at $(\mathbf{R}_x,\mathbf{C}_x)$ derived from (22)

$$d\mathbf{M}_x = -\mathbf{R}_x^{-1}\, d\mathbf{R}_x\mathbf{R}_x^{-1}\mathbf{C}_x + \mathbf{R}_x^{-1}\, d\mathbf{C}_x,$$

$$\begin{aligned}\text{vec}(d\mathbf{M}) &= -(\mathbf{C}_x\mathbf{R}_x^{-T})\otimes\mathbf{R}_x^{-1})\text{vec}(d\mathbf{R}_x)+(\mathbf{I}\otimes\mathbf{R}_x^{-1})\text{vec}(d\mathbf{C}_x)\\ &\stackrel{\text{def}}{=} \mathbf{D}_{M,R}\,\text{vec}(d\mathbf{R}_x)+\mathbf{D}_{M,C}\,\text{vec}(d\mathbf{C}_x),\end{aligned}$$

we obtain from the noncircular Gaussian asymptotic distribution of $(\hat{\mathbf{R}}_x,\hat{\mathbf{C}}_x)$

$$\sqrt{K}(\text{vec}(\hat{\mathbf{R}}_x,\hat{\mathbf{C}}_x)-\text{vec}(\mathbf{R}_x,\mathbf{C}_x)) \xrightarrow{\mathcal{L}} \mathcal{N}_C\left(\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{R}_R & \mathbf{R}_{R,C}\\ \mathbf{R}_{R,C}^H & \mathbf{R}_C\end{pmatrix}, \begin{pmatrix}\mathbf{C}_R & \mathbf{C}_{R,C}\\ \mathbf{C}_{R,C}^T & \mathbf{C}_C\end{pmatrix}\right),$$

whose expressions of $\mathbf{R}_R$, $\mathbf{R}_{R,C}$, $\mathbf{C}_R$ and $\mathbf{C}_{R,C}$ are given in [11], the following convergence in distribution by the standard theorem of continuity (see e.g., [17, Theorem A, p. 122])

$$\sqrt{K}(\text{vec}(\hat{\mathbf{M}})-\text{vec}(\mathbf{M})) \xrightarrow{\mathcal{L}} \mathcal{N}_C(\mathbf{0},\mathbf{R}_M,\mathbf{C}_M),$$

with

$$\mathbf{R}_M = (\mathbf{D}_{M,R},\mathbf{D}_{M,C})\begin{pmatrix}\mathbf{R}_R & \mathbf{R}_{R,C}\\ \mathbf{R}_{R,C}^H & \mathbf{R}_C\end{pmatrix}\begin{pmatrix}\mathbf{D}_{M,R}^H\\ \mathbf{D}_{M,C}^H\end{pmatrix},$$

$$\mathbf{C}_M = (\mathbf{D}_{M,R},\mathbf{D}_{M,C})\begin{pmatrix}\mathbf{C}_R & \mathbf{C}_{R,C}\\ \mathbf{C}_{R,C}^T & \mathbf{C}_C\end{pmatrix}\begin{pmatrix}\mathbf{D}_{M,R}^T\\ \mathbf{D}_{M,C}^T\end{pmatrix}.$$

Then consider the differential of the mapping $\hat{\mathbf{M}}_x \mapsto \hat{\Sigma}_x = \hat{\mathbf{M}}_x^*\hat{\mathbf{M}}_x$ at $\mathbf{M}_x$

$$d\Sigma_x = \mathbf{M}_x^*\, d\mathbf{M}_x + d\mathbf{M}_x^*\mathbf{M}_x,$$

$$\begin{aligned}\text{vec}(d\Sigma) &= (\mathbf{I}\otimes\mathbf{M}_x^*)\text{vec}(d\mathbf{M}_x))+(\mathbf{M}_x\otimes\mathbf{I})\text{vec}(d\mathbf{M}_x^*)\\ &\stackrel{\text{def}}{=} \mathbf{D}_{\Sigma,M}\text{vec}(d\mathbf{M}_x)+\mathbf{D}_{\Sigma,M^*}\text{vec}(d\mathbf{M}_x^*),\end{aligned}$$

which gives the following asymptotic distribution:

$$\sqrt{K}(\text{vec}(\hat{\Sigma}_x)-\text{vec}(\Sigma_x)) \xrightarrow{\mathcal{L}} \mathcal{N}_C(\mathbf{0},\mathbf{R}_{\Sigma_x},\mathbf{C}_{\Sigma_x}),$$

with

$$\mathbf{R}_{\Sigma_x} = (\mathbf{D}_{\Sigma,M},\mathbf{D}_{\Sigma,M^*})\begin{pmatrix}\mathbf{R}_M & \mathbf{C}_M\\ \mathbf{C}_M^* & \mathbf{R}_M^*\end{pmatrix}\begin{pmatrix}\mathbf{D}_{\Sigma,M}^H\\ \mathbf{D}_{\Sigma,M^*}^H\end{pmatrix},$$

[11] Where $(\mathbf{Q}_x)_{i+(j-1)K,\kappa+(l-1)K} = \text{cum}(x_{k,i},x_{k,j},x_{k,\kappa}^*,x_{k,l}^*)$ and $(\mathbf{Q}'_x)_{i+(j-1)K,\kappa+(l-1)K} = \text{cum}(x_{k,i},x_{k,j},x_{k,\kappa},x_{k,l})$ with $\mathbf{x}_k = (x_{k,1},\ldots,x_{k,N})^T$.

$$\mathbf{C}_{\Sigma_x} = (\mathbf{D}_{\Sigma,M}, \mathbf{D}_{\Sigma,M^*}) \begin{pmatrix} \mathbf{C}_M & \mathbf{R}_M \\ \mathbf{R}_M^T & \mathbf{C}_M^* \end{pmatrix} \begin{pmatrix} \mathbf{D}_{\Sigma,M}^T \\ \mathbf{D}_{\Sigma,M^*}^T \end{pmatrix}.$$

Finally, considering the differential of the mapping $\hat{\Sigma}_x \mapsto \ell(\mathbf{x},K) = \det[\mathbf{I} - \hat{\Sigma}_x]$ at Σ_x

$$\begin{aligned} d\ell &= -\det[\mathbf{I} - \Sigma_x]\mathrm{Tr}[(\mathbf{I} - \Sigma_x)^{-1} d\Sigma_x] \\ &= -\det[\mathbf{I} - \Sigma_x]\mathrm{vec}^T((\mathbf{I} - \Sigma_x^T)^{-1})\mathrm{vec}(d\Sigma_x) \overset{\text{def}}{=} \mathbf{D}_{l,\Sigma}\, \mathrm{vec}(d\Sigma_x) \end{aligned}$$

from [24, Theorem 1, p. 149], the convergence in distribution (11) follows with $\sigma_1^2 = \mathbf{D}_{l,\Sigma}\mathbf{C}_{\Sigma_x}\mathbf{D}_{l,\Sigma}^T = \mathbf{D}_{l,\Sigma}\mathbf{R}_{\Sigma_x}\mathbf{D}_{l,\Sigma}^H$ and $\ell_1 = \det[\mathbf{I} - (\mathbf{R}_x^{-1}\mathbf{C}_x)^* \mathbf{R}_x^{-1}\mathbf{C}_x] < 1$ derived from (1). □

Proof of Result 4. To derive the asymptotic distribution of the GLR and then to extend the results of Section 3.1, we replace the classical central limit theorem with the Lyapunov theorem (see e.g., [21, Theorem 2.7.1]) by checking that the Lyapunov conditions (13) are satisfied for the sequence of zero-mean RVs $|x_k^2| - r_{x,k}$ and $x_k^2 - c_{x,k}$. In fact the Lyapunov theorem[12] is valid for zero-mean real-valued scalar RVs u_k. To extend it to the zero-mean complex-valued multidimensional RV $(|x_k^2| - r_{x,k}, x_k^2 - c_{x,k})$, we must elaborate a little bit. First, the extension of the Lyapunov theorem to zero-mean real-valued multidimensional RVs $\mathbf{u}_k$ is straightforward by the application of the Cramer–Wold theorem [21, Theorem 5.1.8] for which the sequence $\mathbf{R}_{u,K}^{-1/2}\sum_{k=1}^K \mathbf{u}_k$ converges in distribution to a zero-mean, Gaussian distribution $\mathcal{N}_R(\mathbf{0},\mathbf{I})$ where $\mathbf{R}_{u,K}^{1/2}$ is an arbitrary square root of $\mathbf{R}_{u,K} \overset{\text{def}}{=} \sum_{k=1}^K E(\mathbf{u}_k\mathbf{u}_k^T)$. Then the Lyapunov theorem applies to the zero-mean complex-valued multidimensional RV $(|x_k^2| - r_{x,k}, x_k^2 - c_{x,k})$, due to isomorphism between $\mathbb{C}$ and $\mathbb{R}^2$. Here, using [25, Theorem 1], there exists a sequence of 2×2 matrices $\mathbf{A}_K$ such that

$$\mathbf{A}_K^{-1}\begin{pmatrix} \hat{r}_x - \overline{r}_{x,K} \\ \hat{c}_x - \overline{c}_{x,K} \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}_C(\mathbf{0},\mathbf{I},\mathbf{\Delta}),$$

with $\mathbf{\Delta}$ is diagonal such that

$$\mathbf{A}_K\mathbf{A}_K^H = \begin{pmatrix} E|\hat{r}_x - \overline{r}_{x,K}|^2 & E(\hat{r}_x - \overline{r}_{x,K})(\hat{c}_x - \overline{c}_{x,K})^* \\ E(\hat{c}_x - \overline{c}_{x,K})(\hat{r}_x - \overline{r}_{x,K})^* & E|\hat{c}_x - \overline{c}_{x,K}|^2 \end{pmatrix},$$

$$\mathbf{A}_K\mathbf{\Delta}\mathbf{A}_K^T = \begin{pmatrix} E(\hat{r}_x - \overline{r}_{x,K})^2 & E(\hat{r}_x - \overline{r}_{x,K})(\hat{c}_x - \overline{c}_{x,K}) \\ E(\hat{c}_x - \overline{c}_{x,K})(\hat{r}_x - \overline{r}_{x,K}) & E(\hat{c}_x - \overline{c}_{x,K})^2 \end{pmatrix},$$

where the terms of those two matrices are given by

$$E(\hat{r}_x - \overline{r}_{x,K})^2 = \frac{1}{K^2}\sum_{k=1}^K (\mathrm{cum}_{x,k} + |c_k|^2 + r_k^2),$$

$$E|\hat{c}_x - \overline{c}_{x,K}|^2 = \frac{1}{K^2}\sum_{k=1}^K (\mathrm{cum}_{x,k} + 2r_k^2),$$

$$E(\hat{c}_x - \overline{c}_{x,K})^2 = \frac{1}{K^2}\sum_{k=1}^K (\mathrm{cum}'_{x,k} + 2c_k^2),$$

$$E(\hat{c}_x - \overline{c}_{x,K})(\hat{r}_x - \overline{r}_{x,K}) = \frac{1}{K^2}\sum_{k=1}^K (\mathrm{cum}''_{x,k} + 2c_k r_k).$$

Under H_0 where the moments of $(x_k)_{k=1,\ldots,K}$ are circular up to the fourth-order, $c_k = 0$, $\mathrm{cum}'_{x,k} = 0$ and $\mathrm{cum}''_{x,k} = 0$, the delta method [21, Chapter 2] derived from the standard theorem of continuity applied to the mapping (18) with the associated differential $dm = -(c/r^2)dr + (1/r)dc$ gives here $dm = (1/r)dc$ and after straightforward algebraic manipulations

$$\sqrt{\frac{K}{\frac{1}{(\overline{r}_{x,K})^2}\frac{1}{K}\sum_{k=1}^K\left(r_{x,k}^2 + \frac{1}{2}\mathrm{cum}_{x,k}\right)}}\left(\hat{m}_x - \frac{\overline{c}_{x,K}}{\overline{r}_{x,K}}\right) \xrightarrow{\mathcal{L}} \mathcal{N}_C(0,1,0).$$

With $\gamma_{x,K} \overset{\text{def}}{=} |\overline{c}_{x,K}|/\overline{r}_{x,K}$, which is the time-averaged circularity coefficient, (14) is proved. □

In the same way, under H_1, (15) is derived from the steps of the Appendix of [11] from the delta method using the two associated differentials

$$dm = -\frac{c}{r^2}dr + \frac{1}{r}dc \quad \text{and} \quad d\gamma = \frac{1}{2\gamma}(m^* dm + m dm^*),$$

$$E\left|\hat{m}_x - \frac{\overline{c}_{x,K}}{\overline{r}_{x,K}}\right|^2 = \left(-\frac{\overline{c}_{x,K}}{\overline{r}_{x,K}^2} \quad \frac{1}{\overline{r}_{x,K}}\right)\mathbf{A}_K\mathbf{A}_K^H\begin{pmatrix} -\frac{\overline{c}_{x,K}^*}{\overline{r}_{x,K}^2} \\ \frac{1}{\overline{r}_{x,K}} \end{pmatrix} + o\left(\frac{1}{K}\right) \overset{\text{def}}{=} r_{m,K} + o\left(\frac{1}{K}\right),$$

$$E\left(\hat{m}_x - \frac{\overline{c}_{x,K}}{\overline{r}_{x,K}}\right)^2 = \left(-\frac{\overline{c}_{x,K}}{\overline{r}_{x,K}^2} \quad \frac{1}{\overline{r}_{x,K}}\right)\mathbf{A}_K\Delta\mathbf{A}_K^T\begin{pmatrix} -\frac{\overline{c}_{x,K}}{\overline{r}_{x,K}^2} \\ \frac{1}{\overline{r}_{x,K}} \end{pmatrix} + o\left(\frac{1}{K}\right) \overset{\text{def}}{=} c_{m,K} + o\left(\frac{1}{K}\right).$$

Then (15) follows with $\sigma_{\gamma,K}$ is given by

$$\sigma_{\gamma,K}^2 = \frac{1}{4\gamma_{x,K}^2}\left(\frac{\overline{c}_{x,K}^*}{\overline{r}_{x,K}^*} \quad \frac{\overline{c}_{x,K}}{\overline{r}_{x,K}}\right)\begin{pmatrix} r_{m,K} & c_{m,K} \\ c_{m,K}^* & r_{m,K}^* \end{pmatrix}\begin{pmatrix} \frac{\overline{c}_{x,K}}{\overline{r}_{x,K}} \\ \frac{\overline{c}_{x,K}^*}{\overline{r}_{x,K}^*} \end{pmatrix}. \qquad \square \tag{23}$$

[12] That we restate for the ease of the reader. If u_k is a sequence of zero-mean scalar real-valued RVs that satisfies $\lim_{K\to\infty}\sum_{k=1}^K E|u_k|^3 / \left(\sqrt{\sum_{k=1}^K E(u_k^2)}\right)^3 = 0$, the sequence $\sum_{k=1}^K u_k/\sqrt{\sum_{k=1}^K E(u_k^2)}$ converges in distribution to a zero-mean, unit variance Gaussian distribution $\mathcal{N}_R(0,1)$.

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