PROCEEDINGS OF THE ICASSP, HONG-KONG, 2003

PARTICLE FILTERING WITH PAIRWISE MARKOV PROCESSES

François Desbouvries and Wojciech Pieczynski

Institut National des Télécommunications, Département CITI
9 rue Charles Fourier, 91011 Evry, France
Francois.Desbouvries@int-evry.fr, Wojciech.Pieczynski@int-evry.fr

ABSTRACT

The estimation of an unobservable process $\mathbf{x}$ from an observed process $\mathbf{y}$ is often performed in the framework of Hidden Markov Models (HMM). In the linear Gaussian case, the classical recursive solution is given by the Kalman filter. On the other hand, particle filters are Monte Carlo based methods which provide approximate solutions in more complex situations. In this paper, we consider Pairwise Markov Models (PMM) by assuming that the pair $(\mathbf{x},\mathbf{y})$ is Markovian. We show that this model is strictly more general than the HMM, and yet still enables particle filtering.

1. INTRODUCTION

An important problem in signal processing consists in recursively estimating an unobservable process $\mathbf{x} = \{\mathbf{x}_n\}_{n\in\mathbb{N}}$ from an observed process $\mathbf{y} = \{\mathbf{y}_n\}_{n\in\mathbb{N}}$. HMM, which means that the hidden process $\mathbf{x}$ is Markovian, are widely used to model the stochastic interactions between $\mathbf{x}$ and $\mathbf{y}$.

Let $p(\mathbf{x}_n|\mathbf{y}_{0:n})$ denote the probability density function (pdf) of $\mathbf{x}_n$ given $\mathbf{y}_{0:n} = \{\mathbf{y}_i\}_{i=0}^n$. In this paper we deal with the so-called filtering problem, which consists in recursively computing $p(\mathbf{x}_n|\mathbf{y}_{0:n})$ as new observations become available. In the linear Gaussian case the solution is provided by the well known Kalman filter. However, the exact recursive solution is difficult to compute in the general case, and consequently many approximate techniques have been developed. Among them, particle filters [1] [2] [3] [4] are Monte Carlo methods which aim at propagating an approximation of $p(\mathbf{x}_n|\mathbf{y}_{0:n})$; such methods have found many applications (see e.g. [5]) and have proven to be very efficient in practice.

In this paper we propose to extend particle filtering by using the general idea that enabled us to successfully generalize Hidden Markov Fields (HMF) to Pairwise Markov Fields (PMF) [6], Hidden Markov Chains (with discrete hidden process) (HMC) to Pairwise Markov Chains (PMC) [7] [8], and Hidden Markov Trees (HMT) to Pairwise Markov Trees (PMT) [9]. More precisely, it is well known that if $(\mathbf{x},\mathbf{y})$ is a classical HMM, then the pair $(\mathbf{x},\mathbf{y})$ itself is a Markov Chain (MC). Conversely, starting from the assumption that $(\mathbf{x},\mathbf{y})$ is a MC is an alternate point of view which we deal with in this work.

So in this paper, we directly assume that the pair $(\mathbf{x},\mathbf{y})$ is a MC, and we show : (i) that such a PMM is strictly more general than the classical HMM, in which both $\mathbf{x}$ and $(\mathbf{x},\mathbf{y})$ are MC; and yet (ii) that a particle filter solution can still be computed.

This paper is organized as follows. In section 2 we recall the classical HMM dynamical state-space model, as well as the exact recursive solution and the particle filter approximate solution for that model. In section 3 we introduce the PMM and we derive the exact recursive solution as well as the particle filter approximation for this new model. Finally in section 4 we show that PMM are strictly more general than HMM. In particular, we classify the different situations in a hierarchy of embedded models : HMM with independent noise; general HMM, in which the noise samples need not be independent; and general PMM in which $\mathbf{x}$ is not necessarily Markovian.

2. CLASSICAL HIDDEN MARKOV MODELS

Let us consider the following classical stochastic dynamical system :

$$\left\{ \begin{array}{lcl} \mathbf{x}_{n+1} & = & g_n(\mathbf{x}_n, \mathbf{u}_n) \\ \mathbf{y}_n & = & h_n(\mathbf{x}_n, \mathbf{v}_n) \end{array} \right. , \quad (1)$$

in which $g_n(.,.)$ is some (possibly nonlinear) function from $\mathbb{R}^m \times \mathbb{R}^p$ to $\mathbb{R}^m$, $h_n(.,.)$ is some (possibly nonlinear) function from $\mathbb{R}^m \times \mathbb{R}^q$ to $\mathbb{R}^q$, and $\mathbf{u} = \{\mathbf{u}_n\}_{n\in\mathbb{N}}$ and $\mathbf{v} = \{\mathbf{v}_n\}_{n\in\mathbb{N}}$ are zero-mean sequences which are independent, jointly independent and independent of $\mathbf{x}_0$.

Then one can check that the following properties hold :

$$p(\mathbf{x}_{n+1}|\mathbf{x}_{0:n}) = p(\mathbf{x}_{n+1}|\mathbf{x}_n) \, ; \quad (2)$$

$$p(\mathbf{y}_{0:n}|\mathbf{x}_{0:n}) = \prod_{i=0}^{n} p(\mathbf{y}_i|\mathbf{x}_{0:n}) \, ; \quad (3)$$

$$p(\mathbf{y}_i|\mathbf{x}_{0:n}) = p(\mathbf{y}_i|\mathbf{x}_i) \text{ for all } i, \ 0 \leq i \leq n \, . \quad (4)$$

In other words, $\mathbf{x}$ is a MC, and since it is known only through the observed process $\mathbf{y}$, (1) is often refered to as an HMM. In order to avoid possible confusion, and in view of equation (3), model (1) will however be refered to in the sequel as a Hidden Markov Model with Independent Noise (HMM-IN).

Now, from (2) to (4) we get

$$p(\mathbf{x}_n|\mathbf{x}_{0:n-1},\mathbf{y}_{0:n-1}) = p(\mathbf{x}_n|\mathbf{x}_{n-1}) \ , \quad (5)$$
$$p(\mathbf{y}_n|\mathbf{x}_{0:n},\mathbf{y}_{0:n-1}) = p(\mathbf{y}_n|\mathbf{x}_n) \ , \quad (6)$$

and so

$$p(\mathbf{x}_{0:n}|\mathbf{y}_{0:n}) = \frac{p(\mathbf{x}_n|\mathbf{x}_{n-1})p(\mathbf{y}_n|\mathbf{x}_n)}{p(\mathbf{y}_n|\mathbf{y}_{0:n-1})}p(\mathbf{x}_{0:n-1}|\mathbf{y}_{0:n-1}). \quad (7)$$

Consequently, the recursive propagation of the posterior density of $\mathbf{x}_n$ (i.e., the computing from $p(\mathbf{x}_{0:n-1}|\mathbf{y}_{0:n-1})$ to $p(\mathbf{x}_{0:n}|\mathbf{y}_{0:n})$) under model (1) is given by

$$p(\mathbf{x}_n|\mathbf{y}_{0:n}) = \frac{p(\mathbf{y}_n|\mathbf{x}_n)\int p(\mathbf{x}_n|\mathbf{x}_{n-1})p(\mathbf{x}_{n-1}|\mathbf{y}_{0:n-1})d\mathbf{x}_{n-1}}{p(\mathbf{y}_n|\mathbf{y}_{0:n-1})} \quad (8)$$

If (1) is linear and $\mathbf{u}$ and $\mathbf{v}$ are Gaussian, the posterior densities of $\mathbf{x}$ are also Gaussian and are thus described by their means and covariance matrices. Propagating $p(\mathbf{x}_n|\mathbf{y}_{0:n})$ amounts to propagating these parameters, and in this case equation (8) reduces to the Kalman filter. However, in the general case, computing equation (8) is difficult in practice. Consequently a number of approximate, Monte Carlo based methods have been derived. Among them, particle filtering is a sequential Monte Carlo method which aims at recursively computing an approximation of $p(\mathbf{x}_n|\mathbf{y}_{0:n})$.

Let us recall the principle of particle filtering [1] [2] [3] [4]. Assume that at time n we have a discrete random measure which approximates $p(\mathbf{x}_{0:n-1}|\mathbf{y}_{0:n-1})$:

$$p(\mathbf{x}_{0:n-1}|\mathbf{y}_{0:n-1}) \simeq \sum_{i=1}^{N} w_{n-1}^{(i)}\delta(\mathbf{x}_{0:n-1}-\mathbf{x}_{0:n-1}^{(i)}) \ ,$$

in which $w_{n-1}^{(i)} \propto \frac{p(\mathbf{x}_{0:n-1}^{(i)}|\mathbf{y}_{0:n-1})}{q(\mathbf{x}_{0:n-1}^{(i)}|\mathbf{y}_{0:n-1})}$, $\sum_{i=1}^{N} w_{n-1}^{(i)} = 1$, and $\{\mathbf{x}_{0:n-1}^{(i)}\}_{i=1}^{N}$ are samples drawn from the importance function $q(\mathbf{x}_{0:n-1}|\mathbf{y}_{0:n-1})$. Then in particular

$$p(\mathbf{x}_{n-1}|\mathbf{y}_{0:n-1}) \simeq \sum_{i=1}^{N} w_{n-1}^{(i)}\delta(\mathbf{x}_{n-1}-\mathbf{x}_{n-1}^{(i)}) \ .$$

Let us now further assume that the importance function factors as

$$q(\mathbf{x}_{0:n}|\mathbf{y}_{0:n}) = q(\mathbf{x}_{0:n-1}|\mathbf{y}_{0:n-1})q(\mathbf{x}_n|\mathbf{x}_{0:n-1},\mathbf{y}_{0:n}), \quad (9)$$

i.e. that $q(\mathbf{x}_{0:n}|\mathbf{y}_{0:n})$ admits $q(\mathbf{x}_{0:n-1}|\mathbf{y}_{0:n-1})$ as marginal. Let $\{\mathbf{x}_n^{(i)}\}_{i=1}^{N} \sim q(\mathbf{x}_n|\mathbf{x}_{0:n-1}^{(i)},\mathbf{y}_{0:n})$; then $\{[\mathbf{x}_{0:n-1}^{(i)},\mathbf{x}_n^{(i)}]\}_{i=1}^{N}$ are samples from $q(\mathbf{x}_{0:n}|\mathbf{y}_{0:n})$. Furthermore, from (7) and (9) we get

$$\begin{aligned}\frac{p(\mathbf{x}_{0:n}^{(i)}|\mathbf{y}_{0:n})}{q(\mathbf{x}_{0:n}^{(i)}|\mathbf{y}_{0:n})} &= \frac{p(\mathbf{x}_n^{(i)}|\mathbf{x}_{n-1}^{(i)})p(\mathbf{y}_n|\mathbf{x}_n^{(i)})}{p(\mathbf{y}_n|\mathbf{y}_{0:n-1})q(\mathbf{x}_n^{(i)}|\mathbf{x}_{0:n-1}^{(i)},\mathbf{y}_{0:n})} \times \\ &\quad \frac{p(\mathbf{x}_{0:n-1}^{(i)}|\mathbf{y}_{0:n-1})}{q(\mathbf{x}_{0:n-1}^{(i)}|\mathbf{y}_{0:n-1})} \\ &\propto \underbrace{\frac{p(\mathbf{x}_n^{(i)}|\mathbf{x}_{n-1}^{(i)})p(\mathbf{y}_n|\mathbf{x}_n^{(i)})}{q(\mathbf{x}_n^{(i)}|\mathbf{x}_{0:n-1}^{(i)},\mathbf{y}_{0:n})} w_{n-1}^{(i)}}_{\tilde{w}_n^{(i)}} .\end{aligned}$$

Finally, $\sum_{i=1}^{N} w_n^{(i)}\delta(\mathbf{x}_n-\mathbf{x}_n^{(i)})$, in which $w_n^{(i)} = \frac{\tilde{w}_n^{(i)}}{\sum_{i=1}^{N}\tilde{w}_n^{(i)}}$, approximates $p(\mathbf{x}_n|\mathbf{y}_{0:n})$.

3. PAIRWISE MARKOV MODELS

Let us turn back to model (1). Let $\mathbf{z}_n = [\mathbf{x}_n^T, \mathbf{y}_{n-1}^T]^T$ and let $\mathbf{z}_0 = \mathbf{x}_0$. $\mathbf{z}_n$ satisfy

$$\mathbf{z}_{n+1} = G_n(\mathbf{z}_n, \mathbf{w}_n) \quad (10)$$

for some function $G_n(.,.)$, where the random variables $\mathbf{w}_n = [\mathbf{u}_n^T, \mathbf{v}_n^T]^T$ are zero-mean, independent and independent of $\mathbf{x}_0$. As a consequence, the process $\mathbf{z} = \{\mathbf{z}_n\}_{n\in\mathbb{N}}$ is also a MC.

Throughout this section we will thus consider the general PMM (10). As we now see, this model still enables to solve the filtering problem. Since $\mathbf{z}$ is a MC,

$$p(\mathbf{x}_{n+1},\mathbf{y}_n|\mathbf{x}_{0:n},\mathbf{y}_{0:n-1}) = p(\mathbf{x}_{n+1},\mathbf{y}_n|\mathbf{x}_n,\mathbf{y}_{n-1}). \quad (11)$$

So (5) and (6) respectively become

$$p(\mathbf{x}_n|\mathbf{x}_{0:n-1},\mathbf{y}_{0:n-1}) = p(\mathbf{x}_n|\mathbf{x}_{n-1},\mathbf{y}_{n-1},\mathbf{y}_{n-2}) \ , \quad (12)$$
$$p(\mathbf{y}_n|\mathbf{x}_{0:n},\mathbf{y}_{0:n-1}) = p(\mathbf{y}_n|\mathbf{x}_n,\mathbf{y}_{n-1}) \ . \quad (13)$$

Consequently, the recursive propagation of $p(\mathbf{x}_{0:n}|\mathbf{y}_{0:n})$ under model (10) is now given by

$$p(\mathbf{x}_{0:n}|\mathbf{y}_{0:n}) = \frac{p(\mathbf{x}_n|\mathbf{x}_{n-1},\mathbf{y}_{n-1},\mathbf{y}_{n-2})p(\mathbf{y}_n|\mathbf{x}_n,\mathbf{y}_{n-1})}{p(\mathbf{y}_n|\mathbf{y}_{0:n-1})} \times p(\mathbf{x}_{0:n-1}|\mathbf{y}_{0:n-1}). \quad (14)$$

Taking (14) into account, we see that the particle filter for HMM-IN can be easily generalized to the PMM case :

Particle filter for PMM.

For $i = 1, \cdots, N$,

- Draw $\mathbf{x}_n^{(i)} \sim q(\mathbf{x}_n|\mathbf{x}_{0:n-1}^{(i-1)},\mathbf{y}_{0:n})$,
  set $\mathbf{x}_{0:n}^{(i)} = [\mathbf{x}_{0:n-1}^{(i)},\mathbf{x}_n^{(i)}]$;

- Compute the weights

$$\tilde{w}_n^{(i)} = \frac{p(\mathbf{x}_n^{(i)}|\mathbf{x}_{n-1}^{(i)}, \mathbf{y}_{n-1}, \mathbf{y}_{n-2})p(\mathbf{y}_n|\mathbf{x}_n^{(i)}, \mathbf{y}_{n-1})}{q(\mathbf{x}_n^{(i)}|\mathbf{x}_{0:n-1}^{(i)}, \mathbf{y}_{0:n})} \times w_{n-1}^{(i)},$$

$$w_n^{(i)} = \frac{\tilde{w}_n^{(i)}}{\sum_{i=0}^{N} \tilde{w}_n^{(i)}}.$$

Finally, $\sum_{i=1}^{N} w_n^{(i)} \delta(\mathbf{x}_n - \mathbf{x}_n^{(i)})$ approximates $p(\mathbf{x}_n|\mathbf{y}_{0:n})$.

Remarks.

Particle filtering algorithms have already been developed in the framework of some particular HMM which are more general than the classical HMM-IN [10] [11]. In these models, $\mathbf{x}$ is a MC, and next $p(\mathbf{y}|\mathbf{x})$ is designed in such a way that $\mathbf{z}$ remains a MC. On the other hand, our algorithm is valid for any PMM, irrespective of the possible Markovianity of $\mathbf{x}$. Note that for this algorithm such problems as the choice of the importance function or of a resampling strategy are not adressed here due to lack of space. In fact in this paper, we rather choose to focus on the embedding of the different models, and this is the topic of the next section.

4. PAIRWISE MARKOV MODELS VS. HIDDEN MARKOV MODELS

In this section, we aim at making relations between HMM and PMM clearer. As above, a PMM will denote a model in which $\mathbf{z} = (\mathbf{x}, \mathbf{y})$ is a MC, and an HMM a model in which both $\mathbf{z}$ and $\mathbf{x}$ are MC. As we will see from

$$p(\mathbf{z}_{0:n}) = p(\mathbf{y}_{0:n-1}|\mathbf{x}_{0:n})p(\mathbf{x}_{0:n}) , \quad (15)$$

in a PMM the distribution $p(\mathbf{y}_{0:n-1}|\mathbf{x}_{0:n})$ is Markovian, but $\mathbf{x}$ is not necessarily a MC. So, PMM are strictly more general than general HMM, which are strictly more general than HMM-IN.

Let us thus look for conditions under which the marginal process $\mathbf{x}$ of a MC $\mathbf{z} = (\mathbf{x}, \mathbf{y})$ is itself Markovian.

4.1. a global characterization

Let $\mathbf{z}_n = [\mathbf{x}_n^T, \mathbf{y}_{n-1}^T]^T$ (with $\mathbf{z}_0 = \mathbf{x}_0$) be a MC. Then

$$\begin{aligned} p(\mathbf{z}_{0:n}) &= \frac{p(\mathbf{z}_0, \mathbf{z}_1) \cdots p(\mathbf{z}_{n-1}, \mathbf{z}_n)}{p(\mathbf{z}_1) \cdots p(\mathbf{z}_{n-1})} \\ &= \underbrace{\frac{p(\mathbf{y}_0|\mathbf{x}_0, \mathbf{x}_1) \cdots p(\mathbf{y}_{n-2}, \mathbf{y}_{n-1}|\mathbf{x}_{n-1}, \mathbf{x}_n)}{p(\mathbf{y}_0|\mathbf{x}_1) \cdots p(\mathbf{y}_{n-2}|\mathbf{x}_{n-1})}}_{A} \\ &\times \underbrace{\frac{p(\mathbf{x}_0, \mathbf{x}_1) \cdots p(\mathbf{x}_{n-1}, \mathbf{x}_n)}{p(\mathbf{x}_1) \cdots p(\mathbf{x}_{n-1})}}_{B} . \end{aligned} \quad (16)$$

Note that the presence of B in (16) should not be misleading : though (16) always holds, B is not always equal to $p(\mathbf{x}_{0:n})$. Indeed, comparing (16) with (15), we get the following result :

Proposition 1 *Let $\mathbf{z}_n = [\mathbf{x}_n^T, \mathbf{y}_{n-1}^T]^T$ (with $\mathbf{z}_0 = \mathbf{x}_0$) and $\mathbf{z} = \{\mathbf{z}_n\}_{n\in\mathbb{N}}$. Assume that $\mathbf{z}$ is a MC. Then $\mathbf{x} = \{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is a MC if and only if for all $n > 1$,*

$$p(\mathbf{y}_{0:n-1}|\mathbf{x}_{0:n}) = \quad (17)$$

$$\frac{p(\mathbf{y}_0|\mathbf{x}_0, \mathbf{x}_1)p(\mathbf{y}_0, \mathbf{y}_1|\mathbf{x}_1, \mathbf{x}_2) \cdots p(\mathbf{y}_{n-2}, \mathbf{y}_{n-1}|\mathbf{x}_{n-1}, \mathbf{x}_n)}{p(\mathbf{y}_0|\mathbf{x}_1) \cdots p(\mathbf{y}_{n-2}|\mathbf{x}_{n-1})} .$$

On the other hand, as we sill see in the following, (17) does not always hold.

4.2. local necessary and sufficient conditions

Proposition 1 provides a necessary and sufficient condition ensuring that $\mathbf{x}$ is a MC. However, the condition is given in terms of $p(\mathbf{y}_{0:n-1}|\mathbf{x}_{0:n})$ and is thus difficult to handle. On the other hand, $\mathbf{z}$ is a MC, so for all n the pdf of $\mathbf{z}_{0:n}$ is given in terms of $\{p(\mathbf{z}_i, \mathbf{z}_{i+1})\}_{i=0}^{n-1}$. We are thus looking for a condition expressed in terms of these local pdf. The following proposition provides sufficient conditions which can be checked locally in the framework of a dynamic stochastic model (10).

Proposition 2 *Let $\mathbf{z}_n = [\mathbf{x}_n^T, \mathbf{y}_{n-1}^T]^T$ (with $\mathbf{z}_0 = \mathbf{x}_0$) and $\mathbf{z} = \{\mathbf{z}_n\}_{n\in\mathbb{N}}$. Assume that $\mathbf{z}$ is a MC. Further assume that either*

$$\text{for all } n, \quad p(\mathbf{y}_n|\mathbf{x}_{n+1}, \mathbf{x}_{n+2}) = p(\mathbf{y}_n|\mathbf{x}_{n+1}) \ , \quad (18)$$

or

$$\text{for all } n, \quad p(\mathbf{y}_n|\mathbf{x}_{n+1}, \mathbf{x}_n) = p(\mathbf{y}_n|\mathbf{x}_{n+1}) \ . \quad (19)$$

Then $\{\mathbf{x}_n\}_{n\geq 0}$ is a MC.

Proof. From (16), $\mathbf{x}$ is a MC if and only if $\int A d\mathbf{y}_{0:n-1} = 1$, which is ensured under (18) or under (19). ■

Conversely, we are now looking for local conditions implied if $\mathbf{x}$ is Markovian. In the Gaussian case, the following result holds (the proof is omitted for want of space) :

Proposition 3 *Let $\mathbf{z}_n = [\mathbf{x}_n^T, \mathbf{y}_{n-1}^T]^T$ (with $\mathbf{z}_0 = \mathbf{x}_0$) and $\mathbf{z} = \{\mathbf{z}_n\}_{n\in\mathbb{N}}$. Assume that $\mathbf{z}$ is a MC. Further assume that $\mathbf{z}$ is zero-mean and Gaussian, and that $y_n \in \mathbb{R}$ (i.e. that $q = 1$). If $\{\mathbf{x}_n\}_{n\geq 0}$ is a MC, then for all n, either $p(y_n|\mathbf{x}_{n+1}, \mathbf{x}_{n+2}) = p(y_n|\mathbf{x}_{n+1})$, or $p(y_n|\mathbf{x}_{n+1}, \mathbf{x}_n) = p(y_n|\mathbf{x}_{n+1})$.*

4.3. HMM-IN, General HMM, and PMM

Though $p(\mathbf{x}_{0:n})$ is not necessarily Markovian, let us first remark that if $\mathbf{z}$ is a MC, $p(\mathbf{y}_{0:n-1}|\mathbf{x}_{0:n})$ is always Markovian. The following result holds whether $\mathbf{x}$ is a MC or not:

Proposition 4 *Let $\mathbf{z} = \{\mathbf{z}_n\}_{n\in\mathbb{N}}$. Assume that $\mathbf{z}$ is a MC. Then conditionally on $\mathbf{x}_{0:n}$, the variables $\{\mathbf{y}_i\}_{i=0}^{n}$ form a MC. Moreover, for $1 \le i \le n$,*

$$p(\mathbf{y}_i|\mathbf{y}_{0:i-1}, \mathbf{x}_{0:n}) = p(\mathbf{y}_i|\mathbf{y}_{i-1}, \mathbf{x}_{i:n}).$$

Proof. Since $\mathbf{z}$ is a MC,

$$\begin{aligned} p(\mathbf{y}_i|\mathbf{y}_{0:i-1}, \mathbf{x}_{0:n}) &= \frac{\int p(\mathbf{z}_{0:n}) d\mathbf{y}_{i+1:n-1}}{\int p(\mathbf{z}_{0:n}) d\mathbf{y}_{i:n-1}} \\ &= \frac{\frac{p(\mathbf{z}_0,\mathbf{z}_1)\cdots p(\mathbf{z}_{i-1},\mathbf{z}_i)}{p(\mathbf{z}_1)\cdots p(\mathbf{z}_i)} \int p(\mathbf{z}_{i:n}) d\mathbf{y}_{i+1:n-1}}{\frac{p(\mathbf{z}_0,\mathbf{z}_1)\cdots p(\mathbf{z}_{i-1},\mathbf{z}_i)}{p(\mathbf{z}_1)\cdots p(\mathbf{z}_i)} \int p(\mathbf{z}_{i:n}) d\mathbf{y}_{i:n-1}} \\ &= p(\mathbf{y}_i|\mathbf{y}_{i-1}, \mathbf{x}_{i:n}) \ . \end{aligned}$$

■

We are now ready to classify the different models. As we see, PMM encompass different classes of embedded models : classical HMM with independent noise, HMM with more general noise profile, and finally models in which the state process $\mathbf{x}$ is not Markovian. More precisely:

- Let $\mathbf{x}$ be Markovian, and let further (3) and (4) hold. Then one can show that $p(\mathbf{y}_0|\mathbf{x}_0, \mathbf{x}_1) = p(\mathbf{y}_0|\mathbf{x}_0)$, and $p(\mathbf{y}_{i-1}, \mathbf{y}_i|\mathbf{x}_i, \mathbf{x}_{i+1}) = p(\mathbf{y}_{i-1}|\mathbf{x}_i)p(\mathbf{y}_i|\mathbf{x}_i)$ for $1 \le i \le n-1$. Injecting these equations into (3) and (4), we check that (17) is satisfied, as expected.

- On the other hand, there exist models in which $\mathbf{x}$ and $\mathbf{z}$ are Markovian, but (3) and (4) are not satisfied : conditionally on $\{\mathbf{x}_i\}_{i=0}^{n}$, the variables $\{\mathbf{y}_i\}_{i=0}^{n-1}$ form a MC (see Proposition 4), but they need not be independent, and the conditional pdf $p(\mathbf{y}_i|\mathbf{x}_{0:n})$ need not depend on $\mathbf{x}_i$ only.

- Finally, there exist models for which the necessary condition of Proposition 3 is not satisfied (consider for instance the model

  $$\mathbf{z}_{n+1} = \begin{bmatrix} .5 & .1 \\ 1 & 0 \end{bmatrix} \mathbf{z}_n + \mathbf{w}_n,\ p(\mathbf{w}_n) \sim \mathcal{N}(\mathbf{0}, \begin{bmatrix} 1 & .3 \\ .3 & 1 \end{bmatrix})$$

  and $p(\mathbf{x}_0) \sim \mathcal{N}(0,1)$). This shows that we can find PMM for which $\mathbf{x}$ is not a MC, and thus that model (10) is strictly more general than model (1). This wider generality of PMM with respect to HMM could be of interest in some complex physical situations.

5. REFERENCES

[1] J. S. Liu and R. Chen, "Sequential monte carlo methods for dynamic systems," *Journal of the American Statistical Association*, vol. 93, no. 443, pp. 1032–44, September 1998.

[2] A. Doucet, S. J. Godsill, and C. Andrieu, "On sequential monte carlo sampling methods for bayesian filtering," *Statistics and Computing*, vol. 10, pp. 197–208, 2000.

[3] A. Doucet, N. de Freitas, and N. Gordon, Eds., *Sequential Monte Carlo Methods in Practice*, Statistics for Engineering and Information Science. Springer Verlag, New York, 2001.

[4] M. Sanjeev Arulampalam, S. Maskell, N. Gordon, and T. Clapp, "A tutorial on particle filters for online non-linear / non-gaussian bayesian tracking," *IEEE Transactions on Signal Processing*, vol. 50, no. 2, pp. 174–188, February 2002.

[5] P. M. Djuric, J. Zhang, T. Ghirmai, Y. Huang, and J. H. Kotecha, "Applications of particle filtering to communications : A review," in *Proceedings of the EUropean SIgnal Processing COnference (EUSIPCO)*, Toulouse, France, September 2002.

[6] W. Pieczynski and A. N. Tebbache, "Pairwise markov random fields and segmentation of textured images," *Machine Graphics and Vision*, vol. 9, no. 3, pp. 705–718, 2000.

[7] W. Pieczynski, "Pairwise markov chains," *Accepted for publication, IEEE Tr. Pattern Analysis and Machine Intelligence*, 2002.

[8] S. Derrode and W. Pieczynski, "Sar image segmentation using generalized pairwise markov chains," in *Proceedings of SPIE International Symposium on Remote Sensing*, Crete, Greece, September 22-27, 2002.

[9] W. Pieczynski, "Arbres de markov couple - pairwise markov trees," *Comptes Rendus de l'Académie des Sciences - Mathématiques*, vol. 335, pp. 79–82, 2002, Ser. I (*in French*).

[10] O. Capp´e, "Recursive computation of smoothed functionals of hidden markovian processes using a particle approximation," *Monte Carlo Methods and Applications*, vol. 7, no. 1-2, pp. 81–92, 2001.

[11] P. Del Moral and J. Jacod, "Interacting particle filtering with discrete observations," in *Sequential Monte Carlo Methods in Practice*, A. Doucet, N. de Freitas, and N. Gordon, Eds., pp. 43–75. Springer Verlag, 2001.