Sixth International Conference of Computational Methods in Sciences and Engineering, September, 25-30, Hersonissos, Crete, Grece, 2008.

Pairwise And Uniformly Hidden Markov Fields

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Abstract. In Hidden Markov Fields (HMF) models there are two random fields : the hidden Markov field X and the observed field Y. In Pairwise Markov Fields (PMF) models one directly assumes the Markovianity of the couple (X, Y). PMF are more general than HMF; in fact, in PMF X is not necessarily Markovian. The aim of the paper is to provide some necessary and sufficient conditions under which PMF are HMF. We introduce the notion of “uniformly” HMF (UHMF) and we provide a general condition under which a PMF is an UHMF. Some interest of the presented results in the frame of Triplet Markov Fields (TMF) models, in which a third auxiliary random field is added and one considers the Markovianity of (X, U, Y), is also briefly discussed.

Keywords: Uniformly Hidden Markov Fields, Pairwise Markov Fields, Triplet Markov fields.
PACS: 02.50.Ga, 07.05.Pj

INTRODUCTION

Let S be a finite set of pixels. Hidden Markov Field (HMF) model contains two stochastic processes $X=(X_s)_{s\in S}$ and $Y=(Y_s)_{s\in S}$, in which $X=x$ is unobservable - or hidden - and has to be recovered from the observed $Y=y$. Therefore, $Y=y$ can be seen as a noisy version of $X=x$. To simplify, let us temporarily assume that each X_s takes its values in a finite set of classes $\Omega=\{\omega_1,...,\omega_k\}$, whereas each Y_s takes its values in the set of real numbers R. In the classical HMF widely used the field X is a Markov one, and the distribution of Y conditional on X is given by

$$p(y|x)=\prod_{s\in S} p(y_s|x_s)\,, \qquad (1)$$

so that the distribution of $Z=(X,Y)$ is given by

$$p(x,y)=p(x)\prod_{s\in S} p(y_s|x_s)\,, \qquad (2)$$

with $p(x)$ a Gibbs distribution. Since the seminal papers [5, 7] such models have been widely used in image processing and gave, in general, very satisfying results [8, 9, 15]. However, the property (1) is rather a strong hypothesis and, in particular, textures are difficult to model in such context. To take off this hypothesis, a more general model, called Pairwise Markov Fields (PMF), which consists of directly considering the Markovianity of the couple $Z=(X,Y)$, has been proposed in [10]. PMF are more general than HMF; in fact, a HMF is a PMF, when a PMF is not necessarily a HMF. However, as in PMF the conditional distribution $p(x|y)$ remains a Markov one, which makes him as powerful as HMF in the classical Bayesian segmentation problems, Then the following question naturally arises: in a PMF $Z=(X,Y)$, what are the conditions under which X is a Markov field? Such questions have been answered in the case of Pairwise Markov Chains [13] and Pairwise Markov Trees [6, 11]; however, the problem is more difficult in the case of PMF and to our knowledge there is no general results comparable to the results in [1, 11, 13], until now.

UNIFORMLY HIDDEN MARKOV FIELDS

Let $V = (V_s)_{s\in S}$ be a system of neighborhoods and let C be the set of associated cliques. In this paper we will consider the neighborhoods containing the four nearest neighbors; then the cliques are either singletons, or couples of pixels. To simplify, we will omit the singletons. We will say that $Z = (X, Y)$ is a Pairwise Markov Field (PMF) if its distribution is written

$$p(z) \propto \prod_{(s,t)} \varphi(z_s, z_t) \quad (3)$$

with the product taken over the cliques.

For $S' \subset S$, let $Y_{S'} = (Y_s)_{s\in S'}$ be the restriction of Y to S'. We will say that $Z = (X, Y)$ is a "Uniformly" Hidden Markov Field (UHMF) if it is a PMF such that for each $S' \subset S$ the field $(X, Y_{S'})$ is a Markov one. We will say that $Z = (X, Y)$ is a Hidden Markov Field if it is a PMF such that X is a Markov field:

$$p(x) \propto \prod_{(s,t)} \varphi(x_s, x_t) \quad (4)$$

Of course, an UHMF is a HMF, as $X = (X, Y_{S'})$ with $S' = \emptyset$.

Then we can see that the very classical HMF widely used, which is of the form

$$p(x, y) \propto \prod_{(t,s)} \varphi(x_t, x_s) \prod_{s} p(y_s | x_s), \quad (5)$$

is an UHMF. The aim of this paper is to show the converse property: we will see that under mild stationary conditions an UHMF is a classical HMF given by (5).

Remark 1

Let us remark that Uniformly Hidden Markov Fields are very useful when there can be some missing observations. Therefore we have a hidden realization $X = x$, and $Y = y$ is observed in on $S' \subset S$, but is not observed on $S - S'$. As $X = (X, Y_{S'})$ is a Markov field, the distribution $p(x|y_{S'})$ is Markovian, and thus $X = x$ can be searched from $Y_{S'} = y_{S'}$. Therefore an UHMF allows us to search $X = x$ from $Y_{S'} = y_{S'}$ for any $S' \subset S$.

Let us consider the following Lemma

Lemma

Let f, g, h be three real functions on R such that $\int_R f^2(x)h(x)dx = \int_R g^2(x)h(x)dx = \int_R f(x)g(x)h(x)dx$. Then $f = g$.

The proof consists of noticing that $\langle f, g \rangle = \int_R f(x)g(x)h(x)dx$ is a scalar product and that the hypotheses imply $\langle f - g, f - g \rangle = \|f - g\|^2 = 0$.

Let $Z = (X, Y)$ be a PMF. For each $S' \subset S$ we will set $Y_{S'}$ the restriction of Y to S'. We can state the following original result:

Theorem

Let $Z=(X,Y)$ be a stationary PMF locally with the distribution $p(z)\propto\prod_{(s,t)\in C}\varphi(z_s,z_t)$. The three following conditions

(i) For each $S'\subset S$ the field $(X,Y_{S'})$ is Markovian;
(ii) for each $s\in S$, $(X,Y_{S-\{s\}})$ is Markovian;
(iii) φ is of the form $\varphi(x_s,y_s,x_t,y_t)=\psi^*(x_s,x_t)\phi(x_t,y_t)\phi(x_s,y_s)$,

are equivalent.

Proof.

(i) obviously implies (ii).
(ii) implies (iii).

Let $s\in S$ and let t_1, …, t_4 be the four neighbours of s. Let $\overline{C}_s$ be the set of cliques not containing s. The field $(X,Y_{S-\{s\}})$ being Markovian its distribution is a product on cliques. On the other hand, its distribution is $\int_R[\prod_{(u,t)\in C}\varphi(z_u,z_t)]dy_s$. Thus we have $\int_R[\prod_{(u,t)\in C}\varphi(z_u,z_t)]dy_s=[\prod_{(u,t)\in\overline{C}_s}\varphi(z_u,z_t)]\psi(z_{t_1},x_s)\psi(z_{t_2},x_s)\psi(z_{t_3},x_s)\psi(z_{t_4},x_s)$, which gives

$$\int_R[\frac{\varphi(z_{t_1},x_s,y_s)\varphi(z_{t_2},x_s,y_s)\varphi(z_{t_3},x_s,y_s)\varphi(z_{t_4},x_s,y_s)}{\psi(z_{t_1},x_s)\psi(z_{t_2},x_s)\psi(z_{t_3},x_s)\psi(z_{t_4},x_s)}]dy_s=1 \tag{6}$$

Let $f(x_s,y_s)=\frac{\varphi(z_{t_1},x_s,y_s)\varphi(z_{t_2},x_s,y_s)}{\psi(z_{t_1},x_s)\psi(z_{t_2},x_s)}$ and $g(x_s,y_s)=\frac{\varphi(z_{t_3},x_s,y_s)\varphi(z_{t_4},x_s,y_s)}{\psi(z_{t_3},x_s)\psi(z_{t_4},x_s)}$. Applying the Lemma above to f and g we see that the first does not depend on (z_{t_1},z_{t_2}), and the second one does not depend on (z_{t_3},z_{t_4}). This means that their product does nor depend on $(z_{t_1},z_{t_2},z_{t_3},z_{t_4})$. Thus $\prod_{i=1}^{4}\frac{\varphi(z_{t_i},x_s,y_s)}{\psi(z_{t_i},x_s)}=\zeta(x_s,y_s)$, which implies that $\varphi(x_s,y_s,x_t,y_t)=\psi(x_s,x_t,y_t)\phi(x_s,y_s)$. As s and t can be inverted, we also have $\varphi(x_s,y_s,x_t,y_t)=\psi(x_s,x_t,y_s)\phi(x_t,y_t)$, and thus $\psi(x_s,x_t,y_t)\phi(x_s,y_s)=\psi(x_s,x_t,y_s)\phi(x_t,y_t)$. The last equality means that $\frac{\psi(x_s,x_t,y_t)}{\phi(x_t,y_t)}$, which is equal to $\frac{\psi(x_s,x_t,y_s)}{\phi(x_s,y_s)}$, does not depend on y_t, so that $\frac{\psi(x_s,x_t,y_t)}{\phi(x_t,y_t)}=\psi^*(x_s,x_t)$. Finally $\psi(x_s,x_t,y_t)=\psi^*(x_s,x_t)\phi(x_t,y_t)$, and thus $\varphi(x_s,y_s,x_t,y_t)=\psi^*(x_s,x_t)\phi(x_t,y_t)\phi(x_s,y_s)$, which gives (iii).

(iii) implies (i).

We have $p(x,y_S)\propto\prod_{(s,t)\in C}\psi^*(x_s,x_t)\phi(x_t,y_t)\phi(x_s,y_s)=[\prod_{(s,t)\in C}\psi^*(x_s,x_t)][\prod_{t\in S}\phi^*(x_t,y_t)]$. Thus $S'\subset S$ we have $p(x,y_{S'})\propto\prod_{(s,t)\in C}\psi^*_{st}(x_s,x_t)\prod_{t\in S'}\phi^*(x_t,y_t)\prod_{t\in S-S'}[\int_R\phi^*(x_t,y_t)dy_t]$. Setting $\phi^{**}(x_t)=\int_R\phi^*(x_t,y_t)dy_t$, we have $p(x,y_S)\propto\prod_{(s,t)\in C}\psi^*_{st}(x_s,x_t)\prod_{t\in S-S'}\phi^{**}(x_t)\prod_{t\in S'}\phi^*(x_t,y_t)$. This is a Markov distribution, which ends the proof.

TRIPLET MARKOV FIELDS

PMF are linked with the Triplet Markov Fields (TMF). Introduced in [12] and developed in [3], TMF consist of introducing a third random field U and of assuming that the triplet (X,U,Y) is a Markov field. Such a family of models is a very rich one due to the wide possibilities of choosing the field U. Different TMF have been recently applied in real situations and their interest has been showed, in particular, in non stationary image segmentation [2, 14], or in textured image segmentation [2, 4]. Let us also mention that there are links between the triplets Markov models (chains or fields) and the "theory of evidence" [13, 14]. Of course, such a TMF (X,U,Y) can also be considered as being three PMF: $((X,U),Y)$, $(X,(U,Y))$, $(U,(X,Y))$. Then the results of the previous section can be used in larger context and different richer models can be proposed.

More precisely, we said in Remark 1 above that the Uniformly Hidden Markov Fields (X,Y) are very useful when there can be some missing observations. At the same time, the theorem limits the distributions of UHMF (X,Y) to the classical form (5), which is rather restrictive and can be easily critized. Using TMF allows us to get off the limitative form (5) and to propose, in a non exhaustive manner, the following model. Let $V=(X,U)$ be a Markov field with the distribution of the form (4), where X is the searched finite values field and U is an auxiliary finite values field. Then we can consider an UHMF (V,Y), which allows us to search V from Y in any missing observations context and thus, as having V implies having X, which allows us to search X from Y. However, the Markovianity of (X,U) does not imply the Markovianity of X and thus distribution of (X,Y) is more general than the classical distribution (5).

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