CODED MODULATION SCHEMES BASED ON
REED-SOLOMON CODES OF MODERATE LENGTH

Matteo Albanese and Arnaldo Spalvieri

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Principles of multilevel coded modulation

- Integer lattice $\Lambda \in \mathbb{Z}^N$ and partition chain $\Lambda / \Lambda' / \Lambda''$.
- Outer codes $C'(n, k')$ over $\text{GF}(|\Lambda / \Lambda'|)$ and $C''(n, k'')$ over $\text{GF}(|\Lambda' / \Lambda''|)$.

- Resulting scheme $\mathcal{M} \in \mathbb{Z}^{Nn}$

$$\mathcal{M} = C'(n, k') \otimes [\Lambda / \Lambda'] + C''(n, k'') \otimes [\Lambda' / \Lambda''] + (\Lambda'')^n.$$  

- Staged decoding at the receiver $\Rightarrow$ probability of error $P_{\mathcal{M}}(\mathcal{E}) \leq P_{C'}(\mathcal{E}) + P_{C''}(\mathcal{E}) + nP_{\Lambda''}(\mathcal{E})$.

- Reliability measure of symbol $x_i \in \text{GF}(|\Lambda / \Lambda'|)$

$$\mu(x_i) = \| r - s_i \|^2 - \| r - s' \|^2, \quad i = 0, 1, \ldots, |\Lambda / \Lambda'| - 1.$$  

where $r \in \mathbb{R}^N$ is the received point, $s'$ is the closest point in $\Lambda$ and $s_i$ is the closest point in the coset of $\Lambda'$ associated to $x_i$.
Schemes analyzed

\[ \mathcal{M}_1 = \text{seRS}(16, 6) \otimes [E_8/RE_8] + \text{seRS}(16, 14) \otimes [RE_8/2E_8] + 2E_8^{16} \]
\[ \mathcal{M}_2 = \text{seRS}(16, 9) \otimes [E_8/RE_8] + \text{seRS}(16, 15) \otimes [RE_8/2E_8] + 2E_8^{16} \]

where

\[ E_8 = 2\mathbb{Z}^8 + RM(8, 4, 4), \]
\[ RE_8 = 4\mathbb{Z}^8 + 2(8, 7, 2) + (8, 1, 8), \]

<table>
<thead>
<tr>
<th></th>
<th>dimension</th>
<th>(d_{\text{min}}^2)</th>
<th>(\eta) (bit/2D)</th>
<th>(\gamma) (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathcal{M}_1)</td>
<td>128</td>
<td>16</td>
<td>1.75</td>
<td>6.77</td>
</tr>
<tr>
<td>(\mathcal{M}_2)</td>
<td>128</td>
<td>16</td>
<td>1.50</td>
<td>7.53</td>
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Sub-optimal decoding of $\text{RS}(n, k)$ codes over $\text{GF}(q)$

- Algorithms based on the reordering of the received sequence according to a measure of reliability.
  1. Ordered Statistics (OS) decoding: very good performance especially at low and intermediate SNR. High computational complexity and possible performance loss at high SNR.
  2. Generalized Minimum Distance (GMD) decoding: less complex and asymptotically ($\text{SNR} \to \infty$) optimum. Performance loss at intermediate SNR.

- It is possible to either reduce the OS algorithm complexity at the cost of an increased probability of error, or to extend the GMD algorithm to make it more efficient with a greater computational effort.

- The performance of such algorithms can be analyzed.
The OS algorithm

1. After reordering of the received sequence, the columns of the generator matrix are reordered by the same permutation and reduced into systematic form.

2. Encode the \( k \) Most Reliable Symbols (MRS) to determine the first candidate code word (order-0 reprocessing).

3. Change in all possible ways \( i = 1, 2 \ldots, l \) of the \( k \) MRS and encode each new sequence. Generate a list of \( \sum_{i=0}^{l} (q - 1)^i \binom{k}{i} \) candidate code words (order-\( l \) reprocessing).

4. Compute the distance between each candidate code word and the received sequence. Output the most likely code word among those in the list.

*Complexity reduction:* Avoid changes that correspond to less likely sequences, preserving the possibility of computing the probability of error.
Algebraic Error-and-Erasure Decoder (EED): it is possible to erase $f$ coordinates and correct $e$ errors in the remaining coordinates, as long as

$$2e + f \leq d - 1$$

where $d$ is the minimum Hamming distance of the code.

1. Erase $0, 2, \ldots, d - 1$ ($d$ odd) or $1, 3, \ldots, d - 1$ least reliable symbols ($d$ even) and decode each sequence via an EED.

2. Choose the more likely code word among the $\lceil d/2 \rceil = \lceil (n - k + 1)/2 \rceil$ code words in the list.
Extended GMD algorithm

- RS codes are nested, i.e. \(RS(n, k) \supset RS(n, k - l), \ l = 1, 2, \ldots, k - 1\).
  Furthermore any code word \(c \in RS(n, k)\) can be written as
  \[
  c = c' + r_i
  \]
  where \(c' \in RS(n, k - l)\) and \(r_i \in [RS(n, k) / RS(n, k - l)]\)

- The order of the partition is \(M = |RS(n, k) / RS(n, k - l)| = q^l\)

1. Add to the hard-decision sequence a coset leader of the partition.
2. Decode (GMD decoding) the \(RS(n, k - l)\) code.
3. Select the more likely code word in the list of \(q^l \lceil (n - k + l + 1)/2 \rceil\) candidates.
Performance analysis

For both algorithms the probability of error can be upper bounded as

\[ P(\mathcal{E}) \leq P(\mathcal{F}) + P(\mathcal{E}_{ML}) \]

- \( P(\mathcal{E}_{ML}) \) can be computed via standard techniques.
- \( P(\mathcal{F}) \) depends on the algorithm used:
  * OS, order-\( l \) repr.: probability that more then \( l \) erroneous symbols are among the \( k \) MRI symbols.
  * GMD, probability that more then \( (d - 1 - f)/2 \) erroneous symbols are in the \( f \) unerased positions, for any \( f = 0, 2, \ldots, d - 1 \) (\( d \) odd) or \( f = 1, 3, \ldots, d - 1 \) (\( d \) even). (For the extended algorithm just use the bound for the subcode.)
Simulation results - $M_1$, first level, OS

Word Error Rate (WER) vs. $1/4\sigma^2$

First level of $M_1$: seRS(16,6) over $E_8/RE_8$, order-1,-2,-3, and reductions.

- Bounds on $P(\mathcal{E}_{ML})$ tight at intermediate-to-high SNR.
- Order-3 reprocessing (i.e. 70,966 candidates) virtually optimum performance.
- Reduce candidates by a factor 10 ⇒ loss of 0.2 dB at WER = $10^{-4}$.
Simulation results - $\mathcal{M}_2$, first level, GMD

Word Error Rate (WER) vs. $1/4\sigma^2$

- Bounds on $P(\mathcal{F})$ less tight at high SNR.
- GMD is about 0.7 dB away from ML decoding at WER = $10^{-4}$ (4 candidates).
- GMD coset decoding $\Rightarrow$ improvement of 0.25 dB (80 candidates), and 0.5 dB (1,280 candidates).
Simulation results - $\mathcal{M}_2$

- We can approach the sphere lower bound within 0.5 dB with reasonable complexity.
- Especially for small list sizes, GMD appears to be preferable. In the example the same performance is obtained with 1,280 candidates (GMD) instead of 2,350 (OS).
Application to binary codes

Partition of an inner code instead of a signal constellation. The outer codes are extended 16-ary RS codes. \( C_0, C_1 \) and \( C_2 \) are extended BCH codes of length 16.

Code length \( 16 \times 16 = 256 \).
Code rate \( 0.48 \).

Imperfectness (distance from the bound) \( 1.08 \) dB at \( \text{WER} = 10^{-4} \).
Future work

Application of the scheme to MIMO systems for the wireless channel.

- $4 \times 4$ system with rotated versions of $E_8$ (Boutros et al., IT 96) to encode the 4 information streams.
- $2 \times 2$ systems based on $D_4$ (Belfiore and Rekaya, ITW 03), with quaternary outer codes.