On Sum Rates of Joint Multiple Cell-Site Processing

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Introduction

Possible methods for enhancing cellular system performance by joint multi-cell processing are explored

The UpLink channel is a multiple access channel (MAC), and joint processing of the received signals at the cell-sites is considered (no user cooperation)

The DownLink channel is a non-degraded broadcast channel (BC), where joint pre-processing of the signals transmitted from the cell-sites is considered (assuming single user detection)

The MAC-BC duality principle provides the firm information theoretic connections between these seemingly basically different models

Uplink previous results: [Hanly-Whiting ’93], [Wyner ’94], [Shamai-Wyner ’97], [Somekh-Shamai ’00], [Zaidel-Shamai-Verdú ’01], [Somekh-Zaidel-Shamai ’04], [Grant-Hanly-Evans-Muller ’04]

Downlink previous results: [Shamai-Zaidel ’01], [Jafar-Goldsmith ’02], [Jafar-Foschini-Goldsmith ’04], [Dai-Molisch-Poor ’04]
A “Wyner-type” multi-cell model with $M$ cells ordered on a circle

Motivation: symmetry properties, more amenable to analytical analysis, equivalent to linear models for $M \gg 1$

A fully synchronous, optimally coded system is assumed, with cell-sites located at the cells’ boundaries

There are $K$ users in each cell, and a single receive/transmit antenna at each cell-site

Each user “sees” only the two nearest cell-sites

Models a practical “soft-handoff” scenario at the cells’ boundaries
Outline of the Analysis

- Joint multi-cell processing in the uplink and downlink channels is analyzed in terms of the average per-cell sum-rate capacity and spectral efficiency.

- Both non-fading and frequency-flat fading channels are considered.

- For the uplink, analytical expressions for the sum-rate capacities under various conditions are derived for:
  - Intra-cell TDMA scheduling
  - A “Wide-Band” (WB) scheme (all users are active simultaneously utilizing all bandwidth for coding).

- For the downlink channel, assuming individual per-cell power constraints:
  - An analytical expression for the sum-rate capacity is derived for non-fading channels.
  - Flat-fading channels are analyzed through upper and lower bounds.

- The impact of uplink scheduling is demonstrated.

- The impact of restricted downlink cell-site cooperation is also considered.
Uplink Channel - System Model

- Underlying assumptions:
  - Full channel state information is available to the joint multi-cell receiver only
  - Users cannot cooperate their transmissions in any way
  - Discrete baseband representation of the received $M \times 1$ signal vector for an arbitrary time index, is given by

$$y_{ul} = H_M x_{ul} + z_{ul}$$

- $H_{M[M \times K M]}$ - Channel transfer matrix
- $x_{ul[KM \times 1]} \sim \mathcal{N}_c(0, PI_{KM})$ - Transmitted symbols vector
- $z_{ul[M \times 1]} \sim \mathcal{N}_c(0, I_M)$ - Circularly symmetric AWGN vector

- The total intra-cell transmit power is denoted by $\bar{P} \triangleq KP$
Uplink Channel - System Model (Cont’d)

- The $M \times KM$ channel transfer matrix $H_M$:

$$H_M = \begin{pmatrix}
    a_0 & 0 & \ldots & 0 & b_0 \\
    b_1 & a_1 & 0 & \ldots & 0 \\
    0 & b_2 & a_2 & \ddots & \vdots \\
    \vdots & \vdots & \vdots & \ddots & 0 \\
    0 & \ldots & 0 & b_{M-1} & a_{M-1}
\end{pmatrix}$$

- $\{a_{m[1 \times K]}, b_{m[1 \times K]}\}$ - Fading coefficients experienced by the $K$ users of the $m$th and $[(m - 1) \mod M]$th cells, respectively, received at the $m$th cell-site

- i.i.d. fading among different users (ergodic with respect to the time index)

- The first two power moments of the fading distribution are denoted by

$$m_1 \triangleq E\{a_{m,k}\} = E\{b_{m,k}\} ; \quad m_2 \triangleq E\{|a_{m,k}|^2\} = E\{|b_{m,k}|^2\}$$
The ergodic average per-cell sum-rate capacity of the uplink channel is given by

\[ C_{ul}(\bar{P}) = \frac{1}{M} E_{H} \log \det \left( I_{M} + \frac{\bar{P}}{K} H_{M} H_{M}^{\dagger} \right) \]

The corresponding spectral efficiency [bits/sec/Hz] is obtained through the relation

\[ \bar{P} = C_{ul} \left( \frac{E_{b}^{t}}{N_{0}} \right) \frac{E_{b}^{t}}{N_{0}} \]

where \( \frac{E_{b}^{t}}{N_{0}} \) is the transmit \( \frac{E_{b}}{N_{0}} \) and \( C_{ul}(\bar{P}) = C_{ul}(\frac{E_{b}^{t}}{N_{0}}) \)
Uplink Channel - No fading

- In the absence of fading \( a_{m,k} = b_{m,k} = 1, \forall m, k \)

- The matrix \( \frac{\bar{P}}{K} H_M H_M^\dagger \) becomes a circulant matrix with non-zero row entries \((\bar{P}, 2\bar{P}, \bar{P})\)

- The average per-cell sum-rate capacity depends only on the total intra-cell transmit power \( \bar{P} \):
  - Intra-cell TDMA - Single active user per cell, transmitting \( 1/K \) fraction of the time, signaling with power \( \bar{P} = KP \)
  - Wide-Band (WB) - All \( K \) users are simultaneously active, occupying the whole bandwidth, signaling with power \( P \)

\[ \textbf{Proposition 1} \quad \text{The average per-cell sum-rate capacity, in the absence of fading, is given for } M \to \infty \text{ by} \]

\[ C_{ul-nf}(\bar{P}) = \log \left( \frac{1 + 2\bar{P} + \sqrt{1 + 4\bar{P}}}{2} \right) \]
Uplink Channel - Intra-Cell TDMA - Rayleigh Fading

- Intra-cell TDMA scheme in the presence of Rayleigh fading:
  - Infinite circular array \( M \to \infty \)
  - The setup is equivalent to a two tap ISI time variant channel [Narula ’97]

- **Proposition 2**  
The limiting average per-cell sum-rate capacity with intra-cell TDMA scheduling, as \( M \to \infty \), is given by

\[
C_{tdma}(\bar{P}) = \int_{1}^{\infty} \log x \frac{\log e(x)e^{-\frac{x}{\bar{P}}}}{\text{Ei}\left(\frac{1}{\bar{P}}\right) \bar{P}} \, dx ,
\]

where \( \text{Ei}(x) = \int_{1}^{\infty} \frac{\exp(-xt)}{t} \, dt \) is the exponential integral function

- Following [Narula ’97] it can be shown that \( C_{ul-tdma}(\bar{P}) \leq C_{ul-nf}(\bar{P}) \)
Uplink Channel - WB - Fading

- The focus is on $K \gg 1$, while keeping $\bar{P} = KP$ constant
- Applying the SLLN, the $M \times M$ matrix $\frac{\bar{P}}{K} M H M^\dagger$ becomes circulant with non-zero row entries $\left(|m_1|^2 \bar{P}, 2m_2 \bar{P}, |m_1|^2 \bar{P}\right)$

**Proposition 3** The average per-cell sum-rate capacity of the WB scheme ($K \gg 1$), in the presence of fading, is given for $M \to \infty$ by

$$C_{ul-wb}(\bar{P}) = \log \left( \frac{1 + 2\bar{P}m_2 + \sqrt{1 + 4\bar{P}m_2 + 4\bar{P}^2 (m_2^2 - |m_1|^4)}}{2} \right)$$

- The result upper bounds the corresponding sum-rate capacity for any finite $K$ [Somekh-Shamai ’00]
- $m_1 = 0$ is optimum, and assuming $m_2 = 1$ (e.g., Rayleigh fading):

$$C_{ul-wb}(\bar{P}) = \log(1 + 2\bar{P})$$
Uplink Channel - Extreme SNR Regime

**Proposition 4** The uplink channel extreme SNR regimes, in the absence of fading and $M \gg 1$, are characterized by

$$S_0 = \frac{4}{3} ; \quad \frac{E_t}{N_{0 \min}} = \frac{\log_2 2}{2} ; \quad S_\infty = 1 ; \quad L_\infty = 0$$

**Proposition 5** The uplink channel extreme-SNR regimes for Rayleigh-fading channels and intra-cell TDMA scheduling are characterized as $M \to \infty$ by

$$S_0 = 1 ; \quad \frac{E_t}{N_{0 \min}} = \frac{\log_2 2}{2} ; \quad S_\infty = 1 ; \quad L_\infty \approx 0.84$$

**Proposition 6** For Rayleigh fading, $K \gg 1$, and $\forall M \geq 3$, the uplink extreme-SNR regimes with the WB scheme are characterized by

$$S_0 = 2 ; \quad \frac{E_t}{N_{0 \min}} = \frac{\log_2 2}{2} ; \quad S_\infty = 1 ; \quad L_\infty = -1$$
Uplink Average Per-Cell Spectral Efficiency

Uplink average per-cell spectral efficiency plotted as a function of $\frac{E_b}{N_0}$.

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The impact of $K$ on the average per-cell spectral efficiency in Rayleigh fading channels
Downlink System Model

The received $MK \times 1$ signal vector is given by

$$y_{dl} = H_M^\dagger x_{dl} + z_{dl}$$

- $H_M[M \times KM]$ - Channel transfer matrix
- $x_{dl}[M \times 1]$ - The vector of signals transmitted by the $M$ cell-sites. An equal individual per-cell-site power constraint is assumed:

$$E \left\{ x_{dl} x_{dl}^\dagger \right\}_{(m,m)} \leq \bar{P} \forall m$$

- $z_{dl}[MK \times 1] \sim \mathcal{N}_c(0, I_{MK})$ - Circularly symmetric AWGN vector

Full CSI is available to the joint multi-cell transmitter only

The mobile receivers are assumed to be cognizant of their own CSI, and of the employed transmission strategy

**Theorem 1** [Yu-Lan ’04] The sum capacity of the Gaussian multi-antenna broadcast channel with individual per-antenna power constraints

\[
\mathbb{E}\{xx^\dagger\}(n,n) \leq P_n, \text{ is the same as the sum capacity of a dual MAC with a sum power constraint and with a diagonal and “uncertain” noise:}
\]

\[
C_{\text{dl-sum}} = \min_{\Lambda} \max_{\mathcal{D}} \log \frac{\det (HDH^\dagger + \Lambda)}{\det (\Lambda)},
\]

such that \(\mathcal{D}\) and \(\Lambda\) are \(K \times K\) and \(N \times N\) nonnegative diagonal matrices, satisfying \(\text{trace}(\mathcal{D}) \leq 1\) and \(\sum_n P_n [\Lambda](n,n) \leq 1\), respectively.

Theorem 1 can be *directly* applied to the downlink channel of the circular cell-array.
Dual Uplink System Model

The received $M \times 1$ signal vector of the dual uplink channel is given by

$$\tilde{y}_{ul} = H_M \tilde{x}_{ul} + \tilde{z}_{ul}$$

- $H_M[M \times KM]$ - Channel transfer matrix
- $\tilde{x}_{ul}[KM \times 1]$ - The vector of transmitted symbols with constraints:
  - $E \left\{ \tilde{x}_{ul} \tilde{x}_{ul}^\dagger \right\} = D_M = \text{diag}(D_0, D_1, \cdots, D_{M-1})$
  - $\text{trace} (D_M) \leq 1$
- $D_m[K \times K]$ - Nonnegative diagonal matrix, representing the power constraints of the $m$th cell’s users
- $\tilde{z}_{ul}[M \times 1] \sim \mathcal{N}_c(0, \Lambda_M)$ - The circularly symmetric zero mean AWGN
  - Equal individual power constraints $\Rightarrow \bar{P} \text{trace}(\Lambda_M) \leq 1$
Downlink Average Per-Cell Sum-Rate Capacity

Using Theorem 1 the average per-cell sum-rate capacity is given by [Yu-Lan ‘04]:

\[
C_{dl}(\bar{P}) = E_{H_M} \left\{ \frac{1}{M} \min_{\Lambda_M} \max_{\mathcal{D}_M} \log \frac{\det(\mathbf{H}_M \mathcal{D}_M \mathbf{H}_M^\dagger + \Lambda_M)}{\det(\Lambda_M)} \right\}
\]

The optimization is over all nonnegative diagonal matrices:

- \( \mathcal{D}_M [MK \times MK] \), s.t. \( \text{trace}(\mathcal{D}_M) \leq 1 \)
- \( \Lambda_M [M \times M] \), s.t. \( \text{trace}(\Lambda_M) \leq 1/\bar{P} \)

The corresponding spectral efficiency [bits/sec/Hz] is obtained through the relation

\[
\bar{P} = C_{dl}(\frac{E_b^t}{N_0}) \frac{E_b^t}{N_0}
\]

where \( \frac{E_b^t}{N_0} \) is the transmit \( E_b/N_0 \) and \( C_{dl}(\bar{P}) = C_{dl}(\frac{E_b^t}{N_0}) \).
Downlink - No Fading

- For non fading channels $a_{m,k} = b_{m,k} = 1, \forall m, k$
- The channel transfer matrix becomes “block-circulant”
- The dual MAC optimization problem becomes a function of $\{\text{trace}(D_m)\}$
- All intra-cell schemes with equal $\text{trace}(D_m)$ yield the same sum-rate capacity (e.g., intra-cell TDMA)

**Proposition 7** For non-fading channels the average per-cell downlink sum-rate capacity is given for $M \to \infty$ by

$$C_{dl-nf}(\bar{P}) = C_{ul-nf}(\bar{P}) = \log \left( \frac{1 + 2\bar{P} + \sqrt{1 + 4\bar{P}}}{2} \right)$$

- The conclusion holds for any “block-circulant” channel transfer matrix, and for an **overall** power constraint
**Proposition 8** For $K \gg 1$ and $M \geq 3$, the downlink average per-cell sum-rate capacity for Rayleigh fading satisfies

$$\log \left(1 + \bar{P} \left( (1 - \epsilon) \log_e K + 2 \right) \right) \leq C_{dl}(\bar{P}) \leq \log \left(1 + 2\bar{P} \log_e K \right),$$

for appropriately chosen $\epsilon \in (0, 1)$ and while ignoring little orders of $\log_e K$

- **Lower bound:** scheduling by a Threshold Crossing (TC) policy (at the dual-uplink channel)
  - The rate of the TC scheme with $K \gg 1$ upper bounds the achievable rate of the TC scheme with any $K$
  - The lower bound demonstrates a multi-user diversity gain of $\log_e K$ in addition to an array diversity gain of 2

- **Upper bound:** i.i.d. noise, max fading power values incorporated with Hadamard and Jensen inequalities
Proposition 9 The downlink channel extreme-SNR regimes for Rayleigh fading are characterized by ($K \gg 1$)

\[
\frac{\log_e 2}{2 \log_e K} \leq \frac{E_b^t}{N_0 \min} \leq \frac{\log_e 2}{(1 - \epsilon) \log_e K + 2};
\]

\[S_0 = 2;\]

\[S_\infty = 1;\]

\[-1 - \log_2 \log_e K \leq \mathcal{L}_\infty \leq -\log_2 ((1 - \epsilon) \log_e K + 2).\]
Downlink Average Per-Cell Spectral Efficiency

- Downlink average pre-cell spectral efficiency ($K = 100, \epsilon = 0.2$)
The Impact of (Dual) Uplink Scheduling

The impact of (dual) uplink scheduling on the average per-cell spectral efficiency ($K = 100$)
Restricted Cell-Site Cooperation in the Downlink

Underlying assumptions:

- Joint pre-processing for a cluster of no more than $N$ cells
- Multiple reduced complexity transmitters are used (each with full CSI regarding its own cluster)
- It is assumed that the total number of cells $M \gg N$

**Proposition 10** The downlink average per-cell sum-rate capacity, with pre-processing restricted to $N$ cells, is lower bounded for Rayleigh fading by

$$C_{rp,dl}(\bar{P}) \geq \frac{N}{N+1} \log \left(1 + \bar{P} \left((1 - \epsilon) \log_e K + 2\right)\right), \quad 0 < \epsilon < 1.$$  

The high-SNR slope ("pre-log", "multiplexing gain") is lower bounded by

$$S_{\infty rp} \geq \frac{N}{N+1}.$$
Restricted Cell-Site Cooperation in the Downlink (Cont’d)

It can be concluded that restricting joint pre-processing to no more than $N$ cells, reduces the number of degrees of freedom by no more than a factor of $N/(N+1)$

The relatively low performance degradation due to restricted processing is a result of the particular system model (each user only “sees” two cell-sites)

Even when no CSI whatsoever is available, one looses no more than $1/2$ of the total degrees of freedom (inter-cell time sharing). In this case 2-fold diversity factor can be obtained via two-cell-site cooperation in both uplink (each user is received by two cell-sites), and downlink (Alamouti)
Concluding Remarks

The impact of joint multiple cell-site processing was demonstrated here through a simple analytically tractable circular (Wyner-like) multi-cell model.

The model represents a practical “soft handoff” scenario.

Joint (cooperative) multi-cell processing (without spreading or time sharing) eliminates out-of-cell interference, traditionally a limiting factor.

Uplink channel:

- In the absence of fading, a WB scheme and intra-cell TDMA scheduling are equivalent, and the average per-cell sum-rate capacity approaches the sum-capacity of a single isolated cell in the high-SNR regime.
- In flat-fading channels the WB scheme outperforms intra-cell TDMA scheduling.
- Flat-fading enhances performance already for a small $K$ (in the low-SNR regime for $K > 2$).
- Fading produces the array diversity gain factor of 2 for $K \gg 1$. 
Concluding Remarks (Cont’d)

Downlink channel:

- In the absence of fading the downlink and uplink are equivalent.
- For Rayleigh fading and $K \gg 1$, a multi-user diversity gain factor of $\log e K$ is observed, in addition to the array diversity gain factor of 2.
- The gain is due to cooperation, incorporating the available CSI in the transmitting and receiving ends.
- Restricting joint processing to $N$ cells reduces the number of degrees of freedom by no more than a factor of $N/(N + 1)$.

- The uplink-downlink duality principle guarantees the same uplink multiuser diversity features with proper scheduling.
- Joint multi-cell processing is a key tool in enhancing performance of cellular systems.
Proof of Proposition 7

Three observations:

- The minmax optimization problem is a function of the total intra-cell transmit power in the dual MAC ⇒ We particularize to intra-cell TDMA \((K = 1)\)

- The minmax optimization problem is convex in \(\Lambda_M\) and concave in \(D_M\) [Diggavi-Cover ‘01]

- For any circulant matrix \(A_M[M \times M]\), and diagonal matrix \(B_M[M \times M]\), the quantity \(\det(I_M + A_M B_M A_M^\dagger)\) is invariant to cyclic shifts of the diagonal entries of \(B_M\)

Upper Bound:

- Choose \(\Lambda_M = \Lambda_M^0 = \frac{1}{MP}I_M\)

- Let \(D_M^\star\) be a solution to the remaining maximization problem
Proof of Proposition 7 (Cont’d)

Upper Bound (Cont’d):

The corresponding upper bound is unaffected by any cyclic shift of the diagonal entries of $\mathcal{D}_M^*$.

A “mixing” matrix of all $M$ cyclic shifts $\frac{1}{M} \sum_{m=0}^{M-1} \mathcal{D}_M^*(m)$ is also a solution to the maximization problem.

The upper bound is thus maximized by $\mathcal{D}_M^* = \frac{1}{M} I_M$.

Lower Bound:

Choose $\mathcal{D}_M = \mathcal{D}_M^0 = \frac{1}{M} I_M$.

Using similar arguments it follows that $\Lambda_M^* = \frac{1}{MP} I_M$ is a solution to the remaining minimization problem.

The two bounds coincide to $\frac{1}{M} \log \det \left( I_M + \bar{P} H_M H_M^\dagger \right)$.

The proof is completed by applying well known results on the eigenvalues of circulant matrices [Gray ‘72].
Proof of Proposition 8

Lower (Achievable Rate) Bound:

- Choose a particular dual uplink input covariance matrix complying to a “Threshold Crossing” (TC) scheme.

- Only users received at both cell-sites with fade power levels exceeding some constant \( L \) (i.e., \(|a_{m,k}|^2, |b_{m+1,k}|^2 \geq L\)), are allowed to transmit.

- As \( K \to \infty \) the number of active users per cell crystalizes to \( K_0 \triangleq K e^{-2L} \), transmitting at power \( 1/(K_0 M) \).

- \( L \) should be chosen so that \( K_0 \to \infty \) as \( K \to \infty \).

- Let \( K_0 = K e^{-2L} = K^\epsilon \Rightarrow L = \frac{1-\epsilon}{2} \log_e K \), where \( 0 < \epsilon < 1 \).

- \( \Lambda_M = \frac{1}{MP} I_M \) is the solution to the remaining minimization problem (Jensen).

- The result is an upper bound for any finite \( K \) (with TC).
Proof of Proposition 8 (Cont’d)

Upper Bound:
- Choose a particular noise covariance matrix \( \mathbf{\Lambda}_M^0 = \frac{1}{MP} \mathbf{I}_M \)
- Apply the Hadamard inequality to the resulting \( \max \log \det(\cdot) \) problem
- Bound the channel fades by the strongest fading gain (over all intra-cell users) received at each cell-site
- Observe that the maximum of \( K \) i.i.d. \( \chi^2(2) \) distributed random variables behaves like \( f(K) \triangleq \log_e K + O(\log_e \log_e K) \) for \( K \gg 1 \)
  [Sharif-Hassibi ‘05]
- Apply Jensen’s inequality to the resulting bound
Extreme SNR Characterization

The low-SNR regime is characterized through:

\[
C_{ul}(\frac{E_b^t}{N_0}) \approx S_0/3|_{dB} \left(\frac{E_b^t}{N_0}|_{dB} - \frac{E_b^t}{N_{0\min}}|_{dB}\right)
\]

\[
\frac{E_b^t}{N_{0\min}} \triangleq \log_e 2/\hat{C}_{ul}(0) \quad \text{(Scalar AWGN Ch.: } \frac{E_b^t}{N_{0\min}} = \log_e 2)\]

\[
S_0 \quad \text{(slope)} \triangleq -2 \left[\hat{C}_{ul}(0)\right]^2 / \hat{C}_{ul}(0) \quad \text{(Scalar AWGN Ch.: } S_0 = 2)
\]

\((3|_{dB} = 10 \log_{10} 2, \hat{C}_{ul}(0) \text{ are } \hat{C}_{ul}(0) \text{ the derivative of the capacity in nats/dimension at } \bar{P} = 0)\)

The high-SNR regime is characterized through the affine capacity approximation:

\[
C_{ul}(\bar{P}) \approx \begin{cases} 
S_\infty \left(\frac{\bar{P}|_{dB}}{3|_{dB}} - L_\infty\right) & \text{for } \bar{P} \gg 1 \\
S_\infty (\bar{P}|_{dB} - 3|_{dB}L_\infty) & \text{for } \bar{P} \gg 1
\end{cases}
\]

\[
S_\infty \quad \text{(multiplexing gain)} \triangleq \lim_{\bar{P} \to \infty} \bar{P} \hat{C}_{ul}(\bar{P}) \quad \text{(Scalar AWGN Ch.: } S_\infty = 1)\]

\((\hat{C}_{ul}(\bar{P}) \text{ is the derivative of the capacity in nats/dimension})\)

\[
L_\infty \quad \text{(power offset)} \triangleq \lim_{\bar{P} \to \infty} \left(\log_2 \bar{P} - C_{ul}(\bar{P})/S_\infty\right) \quad \text{(} C_{ul}(\bar{P}) \text{ in bits/dimension)}
\]

\((\text{Scalar AWGN Ch.: } L_\infty = 0)\)
Proof of Propositions 10

Step 1:
- Assume no signals are transmitted to users located in cells indexed by integer multiples of $N + 1 \Rightarrow$ Lower bound
- This generates clusters of $N$ cells with $N + 1$ cell-sites
- No inter-cluster interference
- Each cluster can be viewed as a linear cell-array
- Consider the dual uplink following [Yu-Lan ’04]

Step 2:
- Reduce the number of observables in the dual uplink by adding the signal received at the $(N + 1)$th cell-site to the one received at the first cell-site
- This yields a lower bound to the sum-rate capacity (data processing inequality)
Proof of Propositions 10 (Cont’d)

Step 3:
Observe that the resulting sum-rate capacity equals the capacity with full joint pre-processing for a circular array of $N$ cells.

Step 4:
Multiply the result by $N/(N+1)$ to account for the fact that users of the $(N+1)$th cell and its multiples are ignored.